MTH4104 **Introduction to Algebra**

Extra questions for chapter 5 **Spring 2022**

These questions are for you to practice with on your own schedule. You may e.g. want to try some each week as their topics come up in lectures, or use them in exam revision.

5 Matrices

5.1 Defining matrices

I include in this section questions that are about the addition and multiplication operations on matrices, even if they mention the fact that $\mathrm{M}_n(R)$ is a ring.

Question 5.1.1

(a) Write down a *general* element of the ring $\mathrm{M}_2(\mathrm{M}_2(\mathbb{R}))$. (That is, write out the structure of an element, with brackets etc. in the necessary places, using distinct variables to stand in for the real numbers.)

(b) Give the formula for multiplying two elements of $\mathrm{M}_2(\mathrm{M}_2(\mathbb{R}))$.

(c) How is $\mathrm{M}_2(\mathrm{M}_2(\mathbb{R}))$ related to $\mathrm{M}_4(\mathbb{R})$?

5.2 Matrix rings

Question 5.2.1 Find a multiplicative inverse for the matrix

$$\begin{pmatrix} [7]_{13} & [6]_{13} \\ [10]_{13} & [4]_{13} \end{pmatrix}$$

within $\mathrm{M}_2(\mathbb{Z}_{13})$.

Question 5.2.2 Suppose R is a nontrivial ring with identity.

(a) Specify a non-zero 2×2 matrix N with coefficients in R such that $N^2 = 0$.

(b) Using part (a) or otherwise, prove that $\mathrm{M}_2(R)$ does not satisfy the multiplicative inverse law.

Question 5.2.3 Let R be a ring. Let $\alpha, \beta, \gamma, \delta \in R$ and let $A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ and $B = \begin{pmatrix} \gamma & \delta \\ -\delta & \gamma \end{pmatrix}$.

(a) Calculate AB and $A + B$.

(b) Write down a function $\psi : \mathbb{C} \to \mathrm{M}_2(\mathbb{R})$ that satisfies the rules

$$\psi(zw) = \psi(z)\psi(w) \quad \text{and} \quad \psi(z+w) = \psi(z) + \psi(w).$$

(c) Suppose you knew that $\mathrm{M}_2(\mathbb{R})$ was a ring but were unfamiliar with $\mathbb{C}$. Could you use part (b) to help prove that $\mathbb{C}$ is a field?

Question 5.2.4 The point of this question is to illustrate that the rule for inverting matrices using the determinant only works over *commutative* rings. It uses the ring $\mathbb{H}$ of quaternions, so you'll need to refer to the supplementary notes on quaternions on QMPlus.

(a) Compute the square of the matrix of quaternions

$$A = \begin{pmatrix} i & j \\ j & i \end{pmatrix}$$

within $\mathrm{M}_2(\mathbb{H})$. Bear in mind that $\mathbb{H}$ is a noncommutative ring: be careful that you don't accidentally swap the order of factors when multiplying.

(b) Does A have a multiplicative inverse? Why or why not?

(c) Let us denote the entries of A by $a_{i,j}$. Compute the quaternion

$$a_{1,1}a_{2,2} - a_{1,2}a_{2,1}.$$