Queen Mary University of London

MTH4104 **Introduction to Algebra**

Extra questions for chapter 4 **Spring 2021**

These questions are for you to practice with on your own schedule. You may e.g. want to try some each week as their topics come up in lectures, or use them in exam revision.

4 Polynomials

4.1 Defining polynomials

Question 4.1.1 Let K be a skewfield, and let $f, g \in K[x]$ be two nonzero polynomials. Prove that $fg \neq 0$.

4.2 Polynomial rings

Question 4.2.1 Give a counterexample to the multiplicative inverse law for the ring $\mathbb{R}[x]$ of polynomials in x with real coefficients.

Use the properties of degree in question 2 from the week 8 tutorial sheet to prove that your counterexample is valid.

4.3 Roots and factors

Question 4.3.1 Let $f, g \in \mathbb{R}[x]$ be polynomials, with $\deg g > 0$. Suppose that $\deg f = 8$, and $(x-1)^3$ divides f. What can be said about the multiplicity of 1 as a root of f?

Question 4.3.2 Let f and g be polynomials in $\mathbb{K}[x]$, where K is a field.

(a) Prove that if f divides g, then every root of f is also a root of g.

(b) Is the converse true? Justify your answer.

Question 4.3.3 Let $\sim$ be the relation on the polynomial ring $\mathbb{R}[x]$ defined by

$$\sim = \{(p,q) \in \mathbb{R}[x]^2 : q - p = (x^2+1) \cdot r \text{ for some } r \in \mathbb{R}[x]\}.$$

(a) Prove that $\sim$ is an equivalence relation.

(b) Let K be the set of equivalence classes of $\sim$. Define addition and multiplication operations on K, and prove that they are well-defined.

(c) How is K related to $\mathbb{C}$?

Question 4.3.4 Let $f \in \mathbb{R}[x]$ be a polynomial and $\alpha \in \mathbb{R}$ a real number. Prove that α is a root of f of multiplicity at least 2 if and only if α is a root of both f and f', where f' is the derivative of f with respect to x.

The only Calculus facts you should need for this question are the sum and product rules: if $f, g \in \mathbb{R}[x]$, then

$$(f+g)' = f' + g',$$
$$(fg)' = f' \cdot g + f \cdot g'.$$

4.4 Polynomial division

Question 4.4.1

(a) Let $f = t^{12} - 1$ and $g = t^5 - 1$. Using the Euclidean algorithm for polynomials, show that $\gcd(f, g) = t - 1$, and find polynomials x and y in $\mathbb{R}[t]$ such that $fx + gy = t - 1$.

(b) How is your answer to part (a) related to the Euclidean algorithm for integers?

4.5 The Fundamental Theorem of Algebra

Question 4.5.1 Find all complex solutions z to the equation

$$z^8 - 2z^4 + 1 = 0$$

in standard form $z = a + b\mathrm{i}$, and state their multiplicities. Justify your answer.

For another question on this topic, see the end of the extra document "Solving polynomial equations" on QMPlus.