Queen Mary University of London

MTH4104 **Introduction to Algebra**

Extra questions for chapter 1 **Spring 2021**

These questions are for you to practice with on your own schedule. You may e.g. want to try some each week as their topics come up in lectures, or use them in exam revision.

1 Relations

1.1 Ordered pairs and Cartesian product

Question 1.1.1 Write down an equivalence relation on $\{1,2,3,4,5\}$ with exactly three equivalence classes.

Solution One equivalence relation on $X = \{1,2,3,4,5\}$ with three equivalence classes is

$$\{(1,1),(1,4),(2,2),(2,5),(3,3),(4,1),(4,4),(5,2),(5,5)\}.$$

I got this by taking the equivalence relation $\equiv_3$, which has three equivalence classes, and intersecting it with X^2. Since every equivalence class of $\equiv_3$ has a representative inside X, the intersection will still have 3 equivalence classes.

There are of course other examples.

Question 1.1.2 Let X, Y, and Z be sets. Using the definition of $X \times Y$ given in lecture, carefully write down the meanings of the two expressions $(X \times Y) \times Z$ and $X \times (Y \times Z)$. Explain how they are different, and how they differ from the set of ordered triples

$$\{(x,y,z) : x \in X, y \in Y, z \in Z\}.$$

Solution $(X \times Y) \times Z$ is the set

$$\{((x,y),z) : x \in X, y \in Y, z \in Z\}$$

while $X \times (Y \times Z)$ is the set

$$\{(x,(y,z)) : x \in X, y \in Y, z \in Z\}.$$

These are not equal, nor is either of them equal to

$$\{(x,y,z) : x \in X, y \in Y, z \in Z\},$$

since they differ in the position of the brackets inside the nested pairs. The elements of $(X \times Y) \times Z$ are ordered pairs whose *first* member is an ordered pair; those of $X \times (Y \times Z)$ are ordered pairs whose *second* member is an ordered pair; and the elements of the set written out in the question are ordered triples with no nesting.

1.2 Relations

Question 1.2.1 Let X be a finite set. For a natural number $n \geq 1$, let X^n be the set of ordered n-tuples of elements of X. Prove by induction that $|X^n| = |X|^n$.

Solution This is a standard induction, making use of the proposition from lecture about Cartesian product.

The base case is $n = 1$. Then $X^n = X^1$ is the same set as X — or, if you want to be really pedantic, it is the set $\{(x) : x \in X\}$ with extra parentheses around each element, but in any case it has the same size as X. So

$$|X^1| = |X| = |X|^1.$$

For the inductive step, we assume $|X^n| = |X|^n$ and wish to prove $|X^{n+1}| = |X|^{n+1}$. Now, X^{n+1} is the Cartesian product $X^n \times X$ (again, up to pedantry about parentheses), and we proved in lecture that this implies

$$|X^{n+1}| = |X^n| \cdot |X|;$$

the inductive hypothesis finishes the job, since

$$|X^{n+1}| = |X^n| \cdot |X| = |X|^n \cdot |X| = |X|^{n+1}.$$

This completes the proof.

Question 1.2.2 Let $X = \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ and let R be the relation

$$R = \{(\mathrm{A},\mathrm{A}), (\mathrm{A},\mathrm{B}), (\mathrm{A},\mathrm{C}), (\mathrm{B},\mathrm{A}), (\mathrm{C},\mathrm{A})\}$$

on X. Is R reflexive? symmetric? transitive? an equivalence relation? Justify your answers.

Solution *Reflexive?* Recall the definition: R is *reflexive* if

$$\text{for any } x \in X, \text{ it is true that } (x,x) \in R.$$

Think of this as a test that R has to satisfy in order to deserve the label "reflexive".

It is not hard to spot that R is *not* reflexive, because $(\mathrm{B},\mathrm{B}) \notin R$. It doesn't matter that (A,A) is in R; just a single counterexample kills the property.

Symmetric? The relevant definition is now that R is *symmetric* if

$$\text{for any } x,y \in X \text{ such that } (x,y) \in R, \text{ it is also true that } (y,x) \in R.$$

So we apply this test to R.

In this case it is relatively routine to test all possible pairs (x,y) such that (x,y) is in R — there are five of them — and observe that (y,x) is in R in each case as well:

- $x = \mathrm{A}, y = \mathrm{A}$: $(x,y) = (\mathrm{A},\mathrm{A}) \in R$ and $(y,x) = (\mathrm{A},\mathrm{A})$ is also in R. It's the same pair, of course, but that's of no consequence here.
- $x = \mathrm{A}, y = \mathrm{B}$: $(x,y) = (\mathrm{A},\mathrm{B}) \in R$ and $(y,x) = (\mathrm{B},\mathrm{A})$ is also in R.
- $x = \mathrm{B}, y = \mathrm{A}$: $(x,y) = (\mathrm{B},\mathrm{A}) \in R$ and $(y,x) = (\mathrm{A},\mathrm{B})$ is also in R, using the same two elements of R the other way around.
- $x = \mathrm{A}, y = \mathrm{C}$: these last two go the same as the previous two, but with C instead of B. $(x,y) = (\mathrm{A},\mathrm{C}) \in R$ and $(y,x) = (\mathrm{C},\mathrm{A})$ is also in R.
- $x = \mathrm{C}, y = \mathrm{A}$: $(x,y) = (\mathrm{C},\mathrm{A}) \in R$ and $(y,x) = (\mathrm{A},\mathrm{C})$ is also in R.

We conclude that R *is* symmetric.

Transitive? We are now testing for transivity, defined thus: R is *transitive* if

$$\text{for any } x,y,z \in X \text{ such that } (x,y) \in R \text{ and } (y,z) \in R, \text{ it is also true that } (x,z) \in R.$$

We try a few cases: $(x,y) = (\mathrm{A},\mathrm{A})$ and $(y,z) = (\mathrm{A},\mathrm{B})$ is OK since $(x,z) = (\mathrm{A},\mathrm{B}) \in R$. Let's try again: (A,C) and (C,A) is OK since in this case $x = \mathrm{A}, y = \mathrm{C}, z = \mathrm{A}$ and $(x,z) = (\mathrm{A},\mathrm{A}) \in R$. Or how about $(x,y) = (\mathrm{B},\mathrm{A})$ and $(y,z) = (\mathrm{A},\mathrm{A})$? Then $(x,z) = (\mathrm{B},\mathrm{A}) \in R$. Perhaps R really is transitive?

Without systematically checking *all possible* values that x,y,z can take, we cannot be sure, despite the evidence we've accumulated so far. At this point we could apply brute force and test *all* possible values for x,y,z. That is $3^3 = 27$ cases in total, since each of x,y,z can be either A, B or C; but not all such triples will be relevant to the test since for some of them it will not be true that both $(x,y) \in R$ and $(y,z) \in R$.

It turns out there are counterexamples, and perhaps you spotted them sooner: for example,

$$(\mathrm{B},\mathrm{A}) \in R, \quad (\mathrm{A},\mathrm{B}) \in R \quad \text{but} \quad (\mathrm{B},\mathrm{B}) \notin R.$$

Conclusion: R is *not* transitive after all.

Equivalence relation? Since R is not reflexive or transitive, it clearly is not an equivalence relation.

Question 1.2.3 Given two points in $\mathbb{R}^2$, say $p = (x_1,y_1)$, $q = (x_2,y_2)$, recall that the distance between them is

$$d(p,q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Let R be the relation on the set $\mathbb{R}^2$ such that, for $p,q \in \mathbb{R}^2$, the pair (p,q) is in R if and only if $d(p,q)$ is an integer. Is R reflexive? symmetric? transitive? an equivalence relation? Justify your answers.

Solution As before, we have to test each property using the definition. The difference with the previous question is that, whereas there we were able to show the properties true by examining all cases, here there would be infinitely many cases to check, so we must give a proof.

We also have to make sure we don't get confused between single points, which are pairs of coordinates, and pairs *of points*, which are what elements of R are and contain four coordinates in total.

Reflexive? We would like to test whether $(p,p) \in R$ for every point $p \in \mathbb{R}^2$, that is whether $d(p,p) \in \mathbb{Z}$ for all complex numbers z. But

$$d(p,p) = \sqrt{0^2 + 0^2} = 0$$

is always an integer, which proves R *is* reflexive.

Symmetric? Assume that $(p,q) \in R$, that is that $d(p,q)$ equals an integer k. We would like to show that $(q,p) \in R$, that is that $d(q,p)$ is also an integer. But $d(p,q) = d(q,p)$. I don't mind if you stop there and take that fact as obvious, but if you do want to justify it, you would note that $x_1 - x_2 = -(x_2 - x_1)$ and $y_1 - y_2 = -(y_2 - y_1)$ so

$$d(q,p) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d(p,q).$$

This proves that R *is* symmetric.

Transitive? R is *not* transitive. To find a counterexample, what we need to find are points p, q, and r in the plane so that p and q are an integer distance apart, as are q and r, but p and r are not. Three points on a straight line won't work, but for instance a right triangle with sides 1, 1, and $\sqrt{2}$ will. We can use coordinates $p = (0,0)$, $q = (1,0)$, $r = (1,1)$ for such a triangle. Then

$$d(p,q) = \sqrt{1^2 + 0^2} = 1 \in \mathbb{Z}$$

and

$$d(q,r) = \sqrt{0^2 + 1^2} = 1 \in \mathbb{Z}$$

but

$$d(p,r) = \sqrt{1^2 + 1^2} = \sqrt{2} \notin \mathbb{Z}$$

which is the needed counterexample to transitivity.

Equivalence relation? Again, since R is not transitive, it is not an equivalence relation.

Question 1.2.4 Let X be any nonempty set, and let $R = \emptyset$. Then R is the relation on X such that that xRy is always false. Is R reflexive? symmetric? transitive? an equivalence relation? Justify your answers.

Solution I included this question is to make sure you know how the quantifiers "for all" and "there exists" behave when the quantified variable is drawn from the empty set.

Reflexive? This is the easy one. The definition of the reflexive property asks whether $(x,x) \in R$ for every element $x \in X$. Because X is not empty, there is at least one element $x_1 \in X$, but by definition (x_1, x_1) is not in R, so R is *not* reflexive.

Symmetric? Recall: R is symmetric if and only if, for every pair $(x,y) \in R$, it is also true that $(y,x) \in R$. There are no pairs $(x,y) \in R$. And any statement of form "for all elements of the empty set, [such and such holds]" is vacuously *true* – there is no counterexample! Therefore R *is* symmetric.

Transitive? Similar to the last case. R is transitive if and only if, whenever $x,y,z \in X$ are such that $(x,y) \in R$ and $(y,z) \in R$, it is also true that $(x,z) \in R$. There is no way to choose x, y, and z to satisfy the conditions $(x,y) \in R$ and $(y,z) \in R$, so again R *is* transitive, vacuously.

Equivalence relation? Since R is not reflexive, it is not an equivalence relation.

Question 1.2.5 Let X and Z be any two sets, and $f : X \to Z$ any function. Prove that

$$\{(x,y) \in X^2 : f(x) = f(y)\}$$

is an equivalence relation on X.

Solution We must prove reflexivity, symmetry, and transitivity.

Reflexivity. For any $x \in X$, $f(x) = f(x)$ is true, so xRx.

Symmetry. Let $x,y \in X$ satisfy xRy, so $f(x) = f(y)$. Equality is symmetric so this implies $f(y) = f(x)$, which is yRx.

Transitivity. Let $x,y,z \in X$ satisfy xRy and yRz, so $f(x) = f(y)$ and $f(y) = f(z)$. Then

$$f(x) = f(y) = f(z)$$

so xRz.

Question 1.2.6 Let R be the following relation on the set $X = \mathbb{R}$:

$$R = \{(x,y) \in X^2 : xy \geq 0\}.$$

(a) Prove that R is not an equivalence relation.

(b) Make a small change to X to give a new set Y so that

$$S = \{(x,y) \in Y^2 : xy \geq 0\}.$$

is an equivalence relation. Prove that it is an equivalence relation.

Solution (a) R is not an equivalence relation because it is not transitive. We prove this by counterexample. Take $x = -1$, $y = 0$, and $z = 1$. Then $xy = 0 \geq 0$ and $yz = 0 \geq 0$ but $xz = -1 \ngeq 0$, that is, $(x,y) \in R$ and $(y,z) \in R$ but $(x,z) \notin R$.

Note that I did not ask you to comment on reflexivity or symmetry, so my solution does not have to. (However, R is reflexive and symmetric.)

(b) Any valid counterexample for part (a) will share the feature of "switching between" positive and negative numbers by "passing through" zero. This is the behaviour we have to eliminate. So let's leave zero out, and choose $Y := \mathbb{R} \setminus \{0\}$.

Now S is the relation on $Y\mathbb{R} \setminus \{0\}$ defined by

$$S = \{(x,y) \in X^2 : xy \geq 0\}.$$

We will prove that S is an equivalence relation by proving that it is reflexive, symmetric, and transitive.

Reflexive. We must show that $(x,x) \in S$ for all $x \in Y$. For any nonzero real number x we have $x \cdot x = x^2 \geq 0$, so indeed $(x,x) \in S$ as desired.

Symmetric. Let x and y be nonzero real numbers such that $(x,y) \in S$, that is, such that $xy \geq 0$. We must prove that also $(y,x) \in S$, that is, that $yx \geq 0$. But this is immediate from the assumption because $xy = yx$.

Transitive. Let x, y and z be nonzero real numbers such that $(x,y) \in S$ and $(y,z) \in S$, i.e. such that $xy \geq 0$ and $yz \geq 0$. We must prove from these assumptions that $(x,z) \in S$, i.e. that $xz \geq 0$. Since the product of nonnegative numbers is nonnegative, we have

$$xy^2z = (xy)(yz) \geq 0.$$

Now, y^2 is the square of a nonzero real number, so it is strictly positive, and we may divide both sides of the inequality by it. This yields

$$xz = \frac{xy^2z}{y^2} \geq \frac{0}{y^2} = 0,$$

which was the inequality to be proved.

Question 1.2.7 Let X be the set $\{1,2,3,4\}$.

(a) How many different relations are there on X? Justify your answer.

(b) How many different *reflexive* relations are there on X? Justify your answer.

(c) How many different *symmetric* relations are there on X? Justify your answer.

[Hint: Think about a decision tree for making a relation R on X. Go through the pairs in X^2 in order, and at each level of the tree, decide whether or not R contains that pair. When do you get a choice, and when is the decision actually forced?]

Solution (a) The total number of relations on X is the number of subsets of the set $X \times X$, which has cardinality $4 \cdot 4 = 16$. So the answer is 2^{16}.

(b) We would like to count the subsets of $X \times X$ which, as relations, are reflexive. Let R be a reflexive relation, and view it as a subset of $X \times X$. The definition of reflexivity requires that every pair (x,x) for $x \in X$ is an element of R: that is, R must contain the 4 ordered pairs

$$(1,1), (2,2), (3,3), \text{ and } (4,4).$$

However, reflexivity imposes no condition on whether R contains any of the other $16 - 4 = 12$ ordered pairs in $X \times X$, of the form (x,y) where $x \neq y$. Therefore when counting reflexive relations, we allow all of these choices whether R contains these pairs to be made freely. Altogether, a reflexive relation on X is the union of the set $\{(1,1),(2,2),(3,3),(4,4)\}$ and an arbitrary subset of

$$O := \{(x,y) : x,y \in X, x \neq y\}$$

which is the set of all other pairs. The cardinality of O is 12, so it has 2^{12} subsets, and this is also the number of reflexive relations on X.

(c) Now we would like to count the subsets of $X \times X$ that are symmetric. A little reflection on the symmetric property, first, will help.

If $R \subseteq X \times X$ is symmetric and $(x,y) \in R$, the definition of "symmetric" implies directly that $(y,x) \in R$. But what if (x,y) is not in R? In this case (y,x) cannot be in R either, because if it were, the symmetric property used with the variables exchanged implies that (x,y) would be in R! So in fact (x,y) is in R *if and only if* (y,x) is.

Conversely, if (x,y) is in R if and only if (y,x) is for all elements $x,y \in X$, then X is symmetric. So our task is to count the subsets $R \subseteq X \times X$ with this property.

Let's start trying to build R one ordered pair at a time, choosing for each pair whether R contains it or not.

For pairs with first element 1, we have 4 independent choices to make: $(1,1)$ is either in R or not, as is $(1,2)$, as is $(1,3)$, as is $(1,4)$. After this, we have made 4 different binary choices, so there are 2^4 ways to get to this point in the procedure.

How about pairs with first element 2? We don't get to choose anew whether $(2,1)$ is in R, because the answer to this is the same as the answer to whether $(1,2)$ is in R! But we do get to choose whether $(2,2)$ is in R, as well as whether $(2,3)$ is, and whether $(2,4)$ is. In this paragraph we made 3 more binary choices, which there are 2^3 ways to make.

Similarly, when we move on to pairs with first element 3, we don't get to choose whether $(3,1)$ or $(3,2)$ is in R, those being forced by earlier choices, but we do get to make the other 2 choices, about $(3,3)$ and $(3,4)$; there are 2^2 ways to do this. Finally, when looking at pairs starting with 4, the only new choice we get to make is whether $(4,4)$ is in R, which there are 2^1 ways to do.

The choices in each paragraph above are independent, so altogether, the total number of ways to make them is the product

$$2^4 \cdot 2^3 \cdot 2^2 \cdot 2^1 = 2^{10}.$$

This is therefore the total number of symmetric relations on X.

You may wish to work out and prove for yourself the general formula when $X = \{1,2,\ldots,n\}$.

Question 1.2.8 Write down examples of relations that are:

(a) not reflexive, not symmetric, and not transitive;

(b) reflexive, not symmetric, and not transitive;

(c) not reflexive, symmetric, and not transitive;

(d) reflexive, symmetric, and not transitive;

(e) not reflexive, not symmetric, and transitive;

(f) reflexive, not symmetric, and transitive;

(g) not reflexive, symmetric, and transitive;

(h) reflexive, symmetric, and transitive.

Solution I will give minimalist examples, without proof. Needless to say, there are many other examples.
(a) $\{(1,2),(2,3)\}$ on the set $\{1,2,3\}$
(b) $\{(1,1),(1,2),(2,2),(2,3),(3,3)\}$ on the set $\{1,2,3\}$
(c) $\{(1,2),(2,1)\}$ on the set $\{1,2\}$
(d) $\{(1,1),(1,2),(2,1),(2,2),(2,3),(3,2),(3,3)\}$ on the set $\{1,2,3\}$
(e) $\{(1,2)\}$ on the set $\{1,2\}$
(f) $\{(1,1),(1,2),(2,2)\}$ on the set $\{1,2\}$
(g) $\emptyset$ on the set $\{1\}$
(h) $\{(1,1)\}$ on the set $\{1\}$

1.3 Equivalence relations and partitions

Question 1.3.1 Write down a partition of $\mathbb{Z}$ into four parts, exactly two of which are infinite.

Solution There are many answers. One of them is $\{\{a \in \mathbb{Z} : a < 1\}, \{1\}, \{2\}, \{a \in \mathbb{Z} : a > 2\}\}$. I came up with this by first thinking of the two finite parts: the easiest way to make them is with just one element each, like $\{1\}$ and $\{2\}$. Then the remaining integers need to be partitioned into two infinite parts, and they happen to be conveniently separated into smaller and larger.

I picked the numbers 2 and 4 so that there wouldn't be a familiar" partition that does the job. When you see a question that asks you for an example of something, don't be afraid to try to build a new one if you can't think of a familiar one.

Question 1.3.2 Let R be the following relation on $X = \mathbb{R} \setminus \{0\}$:

$$R = \{(x,y) \in X^2 : x = qy \text{ for some rational number } q\}$$

(a) Prove that R is an equivalence relation.

(b) Write down the equivalence class of 1 in R. Simplify your description as much as you can.

(c) If we had instead taken the set X to be $\mathbb{R}$, would the same rule still define an equivalence relation? Justify your answer.

Solution (a) As we saw in the last section of this document, the roadmap for an equivalence relation proof goes a long way. If you know how to plug in definitions so as to work out what to do at each step, then you know how to do the majority of this question.

To prove R is an equivalence relation, we must show it is reflexive, symmetric, and transitive. We take these in turn.

For reflexivity, we would like to show that, for any nonzero real number x, there exists a rational q such that $x = qx$. Well, clearly, $q = 1$ is a rational number that fills the bill.

For symmetry, we need to show that if we are given nonzero reals x and y so that there is a rational q with $x = qy$, then there is another rational r such that $y = rx$. Given the equation $x = qy$, the natural way to obtain an equation of the form $y = (\cdots)$ is to divide through by q. We can do this because q is nonzero (or else x would be zero). That is, we have $y = x/q = x(1/q)$, and this is of the requisite form, $r = 1/q$ being a rational number. This proves R is symmetric.

Lastly, for transitivity, we must show that given nonzero reals x, y, z and rationals q, r so that $x = qy$ and $y = rz$, there is a rational s so that $x = sz$. Again, your algebraic instincts as to how to express x in terms of z should make short work of this: using $y = rz$ to substitute for y in $x = qy$, we get $x = qrz$, in which $s = qr$ is indeed a rational number. So R is transitive.

(b) The equivalence class of 1 is the set of $y \in X$ such that $(1, y)$ is in R. Because R is symmetric, we may turn this around and say that our equivalence class is also the set of $y \in X$ such that $(y, 1) \in R$. This formulation of the equivalence class is particularly easy to read off the definition of R. It is

$$\{y \in X : y = q \cdot 1 \text{ for some rational number } q\} = \mathbb{Q} \setminus \{0\},$$

the set of nonzero rational numbers. (Again, 0 is a rational number but it is excluded because it is not in X.)

(c) The modified relation on the whole set $\mathbb{R}$ would no longer be an equivalence relation. What would go wrong is the symmetry property, as is suggested by a close reading of the proof in part (a) looking for uses of the assumption that elements of X are nonzero: we can no longer take the inverse of the number q. Indeed, to give a counterexample, $(0, 1)$ is in the modified relation but $(1, 0)$ is not, because $0 = 0 \cdot 1$ but there are no rational solutions to $1 = q \cdot 0$.

Question 1.3.3 Give two different examples of an equivalence relation on the set of integers with an infinite number of equivalence classes.

Solution One easy example of an equivalence relation is the equality relation,

$$\text{“}=\text{”} = \{(x, x) : x \in \mathbb{Z}\}.$$

Every element of $\mathbb{Z}$ is in an equivalence class by itself, so there are infinitely many classes.

To come up with another, what we need from the partition point of view is a second way to divide up the integers into infinitely many sets. We've already done the partition where we leave each integer by itself, so this time we have to put some of them together (but not so many that there are only finitely many sets after we're done). For instance, we could just place 1 and 2 in the same part: this yields the relation

$$R = \{(x,x) : x \in \mathbb{Z}\} \cup \{(1,2),(2,1)\}.$$

Or we could put every integer together with its negative, yielding

$$S = \{(x,y) : x \in \mathbb{Z}, y = x \text{ or } y = -x\}.$$

Question 1.3.4 Give an example of an equivalence relation with an infinite number of equivalence classes, each of which is infinite.

Solution We will use the Equivalence Relation Theorem to do this. The equivalence classes of an equivalence relation are just the parts of its corresponding partition. And it's easier to think of how to make a partition that meets conditions like this: any set of disjoint nonempty sets $\{A, B, \ldots\}$ is a partition of something, namely, of the union $A \cup B \cup \cdots$.

So, how can we get an infinite set of disjoint infinite sets? The set of integers $\mathbb{Z}$ is an easy infinite set. Can we make infinitely many "copies" of it: that is, infinitely many sets $A_1, A_2, \ldots$, all of them as big as $\mathbb{Z}$? Can we make these "copies" disjoint?

One way to achieve this is with ordered pairs. Let

$$A_1 := \{(a,1) : a \in \mathbb{Z}\} = \{\ldots,(-1,1),(0,1),(1,1),(2,1),\ldots\},$$
$$A_2 := \{(a,2) : a \in \mathbb{Z}\} = \{\ldots,(-1,2),(0,2),(1,2),(2,2),\ldots\},$$

and in general

$$A_b := \{(a,b) : a \in \mathbb{Z}\}$$

for, say, each integer b (each natural number would've been enough). Then $\{A_b : b \in \mathbb{Z}\}$ is an infinite set of infinite sets.

Now we use the Equivalence Relation Theorem. The corresponding equivalence relation is the relation R on the set $\mathbb{Z}^2 = \bigcup_{b \in \mathbb{Z}} A_b$ defined by

$$R = \{((a,b),(c,d)) \in (\mathbb{Z}^2)^2 : \text{there exists } e \in \mathbb{Z} \text{ such that } (a,b) \in A_e \text{ and } (c,d) \in A_e\}.$$

The condition $(a,b) \in A_e$ is true if and only if $b = e$, and similarly $(c,d) \in A_e$ is true if and only if $d = e$, so this can be simplified to

$$R = \{((a,e),(c,e)) : a,c,e \in \mathbb{Z}\}.$$

Or, in words, the input to R is two pairs of integers, and the output is "true" if the second components of the pairs are the same, and "false" if they're different.

I hope it hasn't confused you to use a set of pairs $\mathbb{Z}^2$ as the "underlying" set X, so that the relation R is a set of pairs *of pairs*! There are other ways I could have built the example above, for example by using sets like $\{x + a : a \in \mathbb{Z}\}$ within $\mathbb{R}$ as my "copies of $\mathbb{Z}$". Or I could've abandoned $\mathbb{Z}$; for example I could've thought geometrically instead and used the horizontal lines in $\mathbb{C}$ (or $\mathbb{R}^2$).

The reason I chose the example I did is to show you another reason why mathematicians used ordered pairs. Ordered pairs are "made for" appending extra data to a mathematical object, and using them to make "different versions of the same object" is an idea you will probably see again in future modules.

Question 1.3.5 True or false: if X is a set and R is a relation on X which is *not* an equivalence relation, then

$$\{[x]_R : x \in X\}$$

is *not* a partition of X. Justify your answer. (That is: if you said "true", give a proof. If you said "false", give a counterexample, and explain why your counterexample has the necessary properties.)

Solution This statement is **false**. Instead of giving my counterexample (which is far from unique!) right away, I will talk about how I got there.

Recall that the definition of $[x]_R$ is

$$[x]_R = \{y \in X : (x,y) \in R\}.$$

This makes sense for any relation R, not just an equivalence relation.

Imagine that we do have an equivalence relation S. We have a procedure to turn it into a partition: for each $x \in X$, we write down the set $[x]_S$; this list of sets contains some duplicates, but after ignoring repetitions, the list of these sets is a partition. Now, is there any way we could change the input to this procedure without changing the output? Yes: we could rearrange the list of sets, or change which ones are repeated.

For example, let S be the equivalence relation which gives the partition $\{\{1\},\{2,3\}\}$. Then S would be

$$\{(1,1),(2,2),(2,3),(3,2),(3,3)\},$$

with the equivalence class of each element being

$$[1]_S = \{1\},$$
$$[2]_S = \{2,3\},$$
$$[3]_S = \{2,3\}.$$

Let's try to make a little change to that by repeating $\{1\}$ instead of $\{2,3\}$. Can we get a relation R such that

$$[1]_R = \{1\},$$
$$[2]_R = \{1\},$$
$$[3]_R = \{2,3\}?$$

Yes: there's exactly one relation and we can read it off of those equations. For example, since

$$[1]_R = \{y \in X : (1,y) \in R\} = \{1\},$$

we have $(1,1) \in R$ (since $y = 1$ is in the set $\{1\}$) but $(1,2) \notin R$ and $(1,3) \notin R$ (since $y = 2$ and $y = 3$ are not). Similarly, $[2]_R = \{1\}$ so $(2,1) \in R$ while $(2,2) \notin R$ and $(2,3) \notin R$; and $[3]_R = \{2,3\}$ so $(3,1) \notin R$ while $(3,2) \in R$ and $(3,3) \in R$. Altogether, we have got

$$R = \{(1,1),(2,1),(3,2),(3,3)\}.$$

Since we were just trying things out above, let's make sure this R is a counterexample. First of all, R is not an equivalence relation, because it's not reflexive: $(2,2)$ is not in R. (In fact it's not symmetric or transitive either, but we only need to show one of these.) Secondly, the way we made R arranged that

$$[1]_R = \{1\},$$
$$[2]_R = \{1\},$$
$$[3]_R = \{2,3\}.$$

Therefore

$$\{[x]_R : x \in X\} = \{[1]_R, [2]_R, [3]_R\} = \{\{1\}, \{2,3\}\},$$

throwing duplicates out of the (outermost) set. This is indeed a partition, as asked.

Question 1.3.6 Let R and S be two equivalence relations on the same set X. Let $T = R \cap S$.

(a) Say what the relation T means in terms of R and S. In other words, complete the sentence:

> Let x and y be elements of S. Then xTy if and only if ______________.

Your completion should contain "xRy" and "xSy".

(b) Prove that $R \cap S$ is also an equivalence relation on X.

(c) Let P be the partition of X associated to R, Q the partition associated to S, and U the partition associated to $R \cap S$. Describe how you would work out what U was if you were given P and Q.

Solution (a) Remember that the notation xTy means (x,y) is an element of T, and $T = R \cap S$ is an intersection. The intersection of two sets R and S is defined to be the set of all elements which are contained in R and in S. Here R and S are relations, but the logic is just the same: a pair (x,y) is in T if and only if it is in R and in S. Writing that using relation notation gives:

> Let x and y be elements of S. Then xTy if and only if xRy and xSy.

(b) To prove that something is an equivalence relation, we have to prove that it is reflexive, symmetric, and transitive, using the definitions. In this case, the relation of interest is $T = R \cap S$, and all we know about it is that it is the intersection of two equivalence relations: that is, (x,y) is in $R \cap S$ if and only if it is in both R and S. So we will have to work abstractly, by parlaying the equivalence relation properties from R and S over to T.

We start with reflexivity. Recall the definition: R is *reflexive* if

$$\text{for any } x \in X, \text{ it is true that } (x,x) \in R.$$

This is true of R and also of S. So for any $x \in X$, we know that $(x,x) \in R$ and $(x,x) \in S$. This implies that $(x,x) \in R \cap S = T$. Therefore T is reflexive.

For symmetry, the relevant definition is that R is *symmetric* if

$$\text{for any } x,y \in X \text{ such that } (x,y) \in R, \text{ it is also true that } (y,x) \in R.$$

We know R and S are symmetric, and must investigate the question for $R \cap S$. So we assume that some pair (x,y) is in T, and must prove that (y,x) is in T as well.

Since the pair (x,y) is an element of $T = R \cap S$, it is also an element of R and of S separately. Both R and S are symmetric, so these last facts imply that (y,x) is an element of R and an element of S. Therefore (y,x) is an element of the intersection T. This proves T is symmetric.

For transitivity, this is the definition: R is *transitive* if

$$\text{for any } x,y,z \in X \text{ such that } (x,y) \in R \text{ and } (y,z) \in R, \text{ it is also true that } (x,z) \in R.$$

The proof here goes along much the same lines as for symmetry. To show T transitive, we may assume (x,y) and (y,z) are in $R \cap S$, and wish to show (x,z) is also. Now since (x,y) and (y,z) are in T, both of these pairs are in R, as well as in S. Because R is transitive, we may conclude $(x,z) \in R$; and because S is transitive we also find that $(x,z) \in S$. Therefore $(x,z) \in T$, being in both R and S individually. This proves that T is transitive.

All in all, we have shown that T is an equivalence relation.

Question 1.3.7 Let $P = \{P_1, P_2, \ldots\}$ and $Q = \{Q_1, Q_2, \ldots\}$ be two partitions of the same set X. Let R be the equivalence relation associated to P, and S the equivalence relation associated to Q.

Suppose that every part P_i is a subset of one of the parts Q_j. What does this imply about R and S? Prove your assertion.

Solution The property of R and S that I was looking for is:

For all x and y in X, if xRy then xSy.

How would you come up with this? This is the sort of question for which I'd recommend drawing diagrams of P and Q. In the diagram, each of the sets Q_j would contain various smaller sets P_i inside it, and each P_i would lie wholly inside one Q_j. (In fact, each Q_j is partitioned by the P_i inside it! But you don't need to notice that.) Now xRy means "x and y are in the same part of P", so draw two dots x and y in one of your sets P_i. Well then, x and y are also clearly inside the same set Q_j, and this is exactly the meaning of xSy. So xRy implies xSy.

This visual reasoning is not a proof, but it's very easily converted to one by smartening it up a bit, as follows.

To prove that for all x and y in X, if xRy then xSy, we must assume that x and y are two elements of X such that xRy holds, and prove that then xSy holds. By the Equivalence Relation Theorem, our assumption xRy implies that there is a part P_i of P such that $x \in P_i$ and $y \in P_i$. We have supposed that there is a part Q_j of Q such that $P_i \subseteq Q_j$, which implies $x \in Q_j$ and $y \in Q_j$. Now using the other direction of the Equivalence Relation Theorem, this implies xSy, which completes the proof.

Question 1.3.8 [Not a fair question! Included for historical interest.] Spot the two "errors" in Figure 1, if it is taken as a list of all partitions of a set of five elements.

Solution The *genjimon* are drawings of the partitions on the set $\{1,2,3,4,5\}$, where the numbers correspond to the five legs at the bottom of each drawing. Two legs are in the same part of the partition if they are connected at top. You should regard plus-shaped crossings as one line going underneath another, not as genuine junctions.

There are only 52 different partitions of the set $\{1,2,3,4,5\}$, so this list must contain some repetitions, which are the "errors" I mean. The repetitions are that *genjimon* 35 and 42 both represent the partition $\{\{1,2,4\},\{3,5\}\}$, and number 54, which is anomalous in joining up some of the legs at the bottom rather than at the top, would seem to represent the same partition $\{\{1,2,3,4,5\}\}$ as number 53.

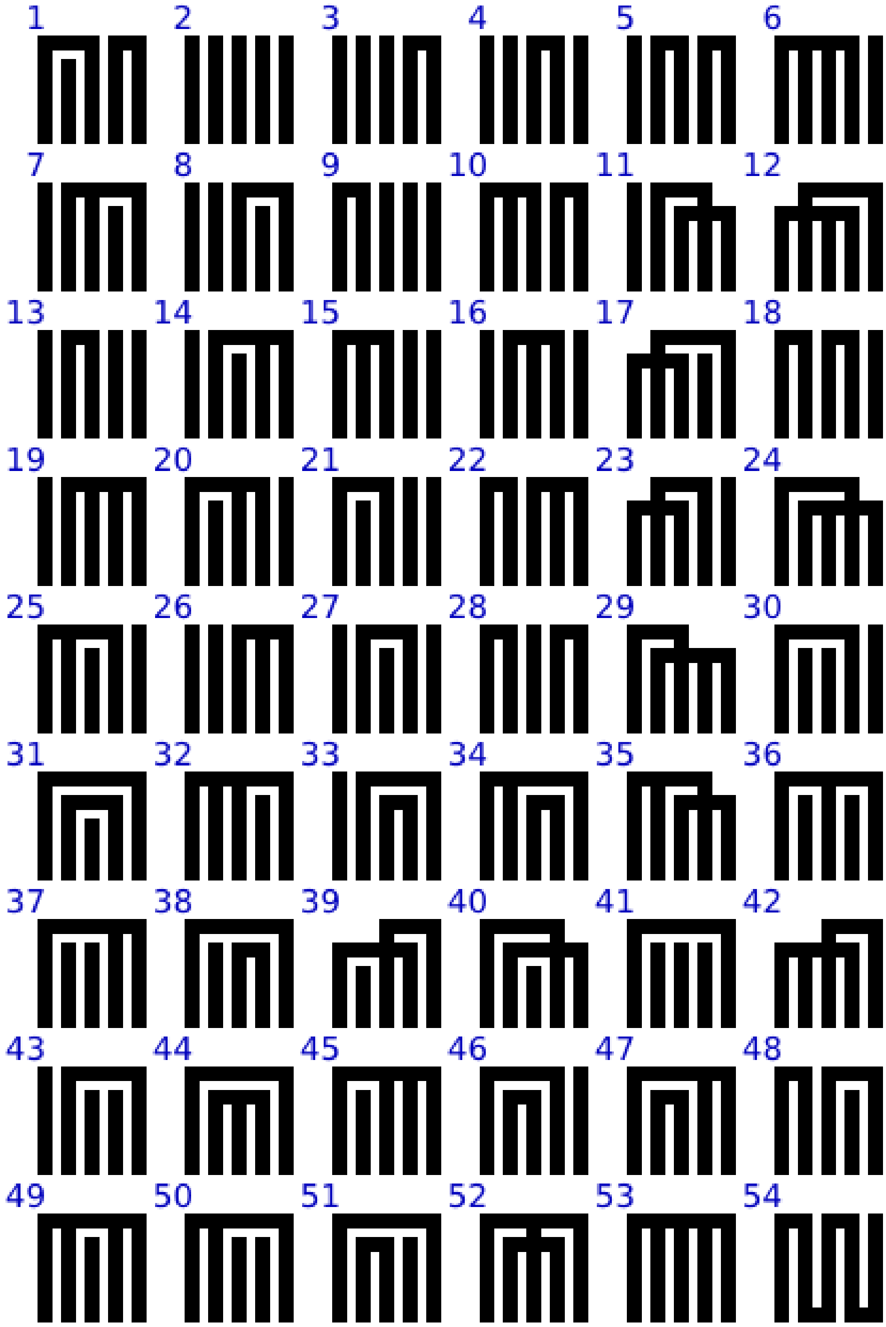

Figure 1: The 54 *genjimon* emblems, traditionally associated with the 54 chapters of the *Tale of Genji* (MURASAKI Shikibu, c. 1008; arguably the world's first novel) and used in *ukiyo-e* prints to identify the subject matter with a particular chapter.
Image credit: altered from a version by Wikipedia user "AnonMoos".