Queen Mary
University of London

MTH4104 **Introduction to Algebra**

Extra questions for chapter 1 **Spring 2021**

These questions are for you to practice with on your own schedule. You may e.g. want to try some each week as their topics come up in lectures, or use them in exam revision.

1 Relations

1.1 Ordered pairs and Cartesian product

Question 1.1.1 Write down an equivalence relation on $\{1,2,3,4,5\}$ with exactly three equivalence classes.

Question 1.1.2 Let X, Y, and Z be sets. Using the definition of $X \times Y$ given in lecture, carefully write down the meanings of the two expressions $(X \times Y) \times Z$ and $X \times (Y \times Z)$. Explain how they are different, and how they differ from the set of ordered triples

$$\{(x,y,z) : x \in X, y \in Y, z \in Z\}.$$

1.2 Relations

Question 1.2.1 Let X be a finite set. For a natural number $n \geq 1$, let X^n be the set of ordered n-tuples of elements of X. Prove by induction that $|X^n| = |X|^n$.

Question 1.2.2 Let $X = \{\mathrm{A},\mathrm{B},\mathrm{C}\}$ and let R be the relation

$$R = \{(\mathrm{A},\mathrm{A}),(\mathrm{A},\mathrm{B}),(\mathrm{A},\mathrm{C}),(\mathrm{B},\mathrm{A}),(\mathrm{C},\mathrm{A})\}$$

on X. Is R reflexive? symmetric? transitive? an equivalence relation? Justify your answers.

Question 1.2.3 Given two points in $\mathbb{R}^2$, say $p = (x_1,y_1)$, $q = (x_2,y_2)$, recall that the distance between them is

$$d(p,q) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}.$$

Let R be the relation on the set $\mathbb{R}^2$ such that, for $p,q \in \mathbb{R}^2$, the pair (p,q) is in R if and only if $d(p,q)$ is an integer. Is R reflexive? symmetric? transitive? an equivalence relation? Justify your answers.

Question 1.2.4 Let X be any nonempty set, and let $R = \emptyset$. Then R is the relation on X such that that xRy is always false. Is R reflexive? symmetric? transitive? an equivalence relation? Justify your answers.

Question 1.2.5 Let X and Z be any two sets, and $f : X \to Z$ any function. Prove that

$$\{(x,y) \in X^2 : f(x) = f(y)\}$$

is an equivalence relation on X.

Question 1.2.6 Let R be the following relation on the set $X = \mathbb{R}$:

$$R = \{(x,y) \in X^2 : xy \geq 0\}.$$

(a) Prove that R is not an equivalence relation.

(b) Make a small change to X to give a new set Y so that

$$S = \{(x,y) \in Y^2 : xy \geq 0\}.$$

is an equivalence relation. Prove that it is an equivalence relation.

Question 1.2.7 Let X be the set $\{1,2,3,4\}$.

(a) How many different relations are there on X? Justify your answer.

(b) How many different *reflexive* relations are there on X? Justify your answer.

(c) How many different *symmetric* relations are there on X? Justify your answer.

[Hint: Think about a decision tree for making a relation R on X. Go through the pairs in X^2 in order, and at each level of the tree, decide whether or not R contains that pair. When do you get a choice, and when is the decision actually forced?]

Question 1.2.8 Write down examples of relations that are:

(a) not reflexive, not symmetric, and not transitive;

(b) reflexive, not symmetric, and not transitive;

(c) not reflexive, symmetric, and not transitive;

(d) reflexive, symmetric, and not transitive;

(e) not reflexive, not symmetric, and transitive;

(f) reflexive, not symmetric, and transitive;

(g) not reflexive, symmetric, and transitive;

(h) reflexive, symmetric, and transitive.

1.3 Equivalence relations and partitions

Question 1.3.1 Write down a partition of $\mathbb{Z}$ into four parts, exactly two of which are infinite.

Question 1.3.2 Let R be the following relation on $X = \mathbb{R} \setminus \{0\}$:

$$R = \{(x, y) \in X^2 : x = qy \text{ for some rational number } q\}$$

(a) Prove that R is an equivalence relation.

(b) Write down the equivalence class of 1 in R. Simplify your description as much as you can.

(c) If we had instead taken the set X to be $\mathbb{R}$, would the same rule still define an equivalence relation? Justify your answer.

Question 1.3.3 Give two different examples of an equivalence relation on the set of integers with an infinite number of equivalence classes.

Question 1.3.4 Give an example of an equivalence relation with an infinite number of equivalence classes, each of which is infinite.

Question 1.3.5 True or false: if X is a set and R is a relation on X which is *not* an equivalence relation, then

$$\{[x]_R : x \in X\}$$

is *not* a partition of X. Justify your answer. (That is: if you said "true", give a proof. If you said "false", give a counterexample, and explain why your counterexample has the necessary properties.)

Question 1.3.6 Let R and S be two equivalence relations on the same set X. Let $T = R \cap S$.

(a) Say what the relation T means in terms of R and S. In other words, complete the sentence:

Let x and y be elements of S. Then xTy if and only if ______________.

Your completion should contain "xRy" and "xSy".

(b) Prove that $R \cap S$ is also an equivalence relation on X.

(c) Let P be the partition of X associated to R, Q the partition associated to S, and U the partition associated to $R \cap S$. Describe how you would work out what U was if you were given P and Q.

Question 1.3.7 Let $P = \{P_1, P_2, \ldots\}$ and $Q = \{Q_1, Q_2, \ldots\}$ be two partitions of the same set X. Let R be the equivalence relation associated to P, and S the equivalence relation associated to Q.

Suppose that every part P_i is a subset of one of the parts Q_j. What does this imply about R and S? Prove your assertion.

Question 1.3.8 [Not a fair question! Included for historical interest.] Spot the two "errors" in Figure 1, if it is taken as a list of all partitions of a set of five elements.

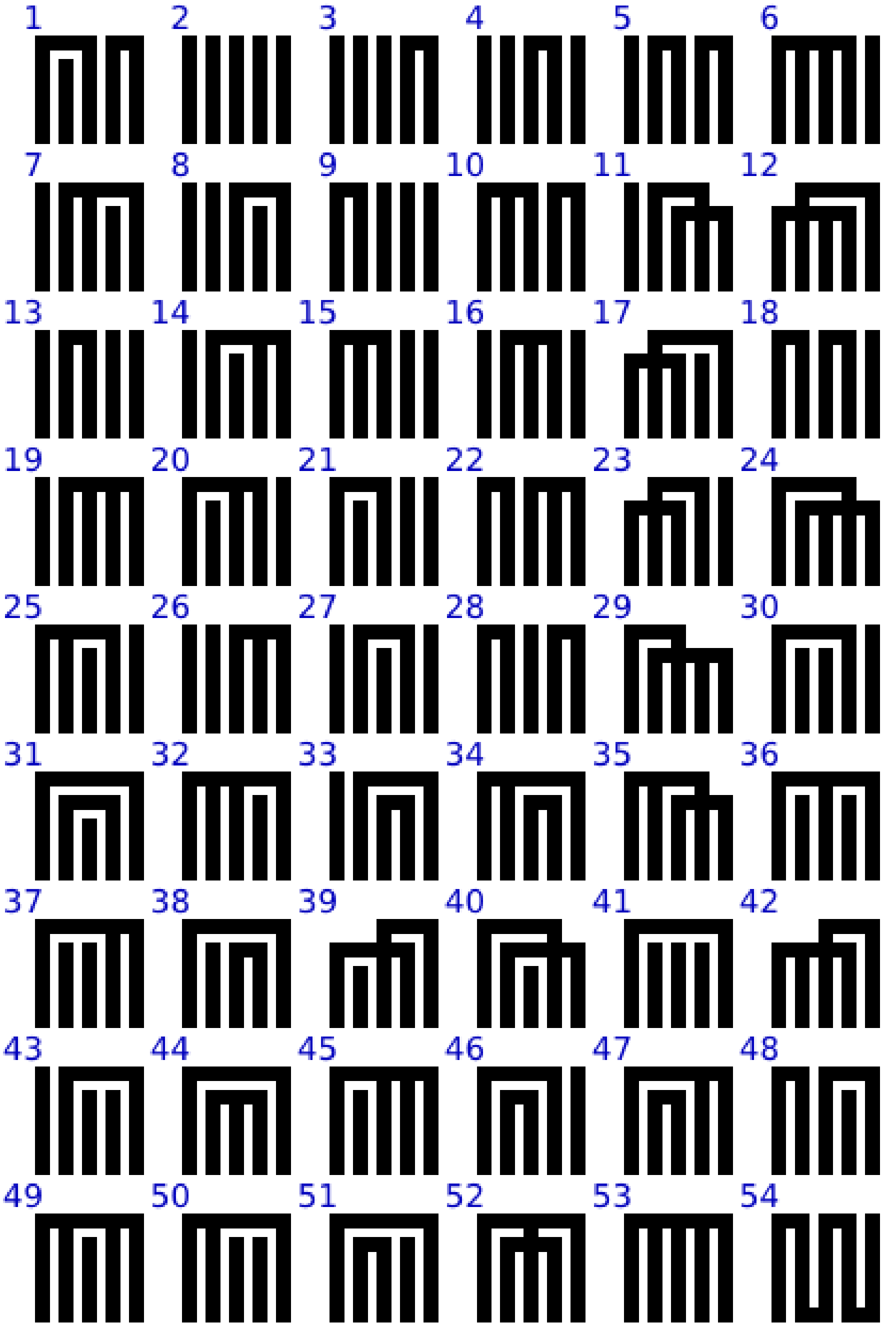

Figure 1: The 54 *genjimon* emblems, traditionally associated with the 54 chapters of the *Tale of Genji* (MURASAKI Shikibu, c. 1008; arguably the world's first novel) and used in *ukiyo-e* prints to identify the subject matter with a particular chapter.
Image credit: altered from a version by Wikipedia user "AnonMoos".