Programme of the 2020 course

Relativistic Waves and Quantum Fields (SPA7018U/P)
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This programme details very precisely the list of the topics we plan to cover in the 2020-2021 academic year, divided by lecture.

Lectures 1–3: Spacetime and its symmetries

- Lorentz transformations: $\Lambda^T \eta \Lambda = \eta$. Definition of group. Proper/improper transformations. Orthochronous transformations. The proper, orthochronous transformations form a subgroup of the Lorentz group. Relativity principle.
- The sign of the time component of a timelike vector is invariant under proper, orthochronous transformations.
• Number of independent parameters of the Lorentz group. Examples of Lorentz transformations: rotations, boosts. Rapidity.
• Covariant and contravariant vectors. Transformation of derivatives $\partial^\mu$ and $\partial_\mu$. Levi-Civita (pseudo)tensor. Invariant quantities.
• Microcausality.
• A cursory look at the Poincaré group. Particles as irreducible representations of the Poincaré group.*
• The potential $A^\mu$ and the field strength $F^{\mu\nu}$. Relativistic form of Maxwell’s equations $\partial_\mu F^{\mu\nu} = J^\nu$. Current conservation. Gauge invariance.

Lectures 4–5: Relativistic free particles and fields

• Lagrangians and the principle of least action (revision). Action for a relativistic free particle. Four-momentum.
• Example 1: the real Klein-Gordon field and its Lagrangian density. Klein Gordon equation.
• Example 2: the complex Klein-Gordon field and its Lagrangian density.

Lectures 6–9: Symmetries and conservation laws

• Total field transformations and infinitesimal variations:
  $$\delta_T \phi = \phi'(x') - \phi(x) = \delta \phi + (\partial_\mu \phi) \delta x^\mu.$$ 
• Conditions for the invariance of the action under a certain transformation. Applications to Lorentz transformations and translations: scalar fields, vector fields and their transformation properties under these transformations.
• Noether’s theorem in classical mechanics (revision).
• Example 1: conserved current associated to $U(1)$ transformations for the complex Klein-Gordon field.
• Example 2: conserved current(s) associated to four-translations. The energy-momentum tensor. Hamiltonian density and momentum density. $\mathcal{H} = T^{00} = \pi(\partial_\mu \phi) - \mathcal{L}$.
• Example 3: conserved current(s) associated to Lorentz transformations, $M^{\mu;\nu} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}$ (homework).

Lectures 10–12: Quantisation of the Klein-Gordon field

• Solution to the field equations. Positive and negative energy modes. 
  $$\phi(x) = \int d^3 k \, N(k) \left[ a(k) e^{-ik \cdot x} + b^\dagger(k) e^{ik \cdot x} \right]$$ with $k_0 = E(k)$. 

Lectures 15–26: The Dirac equation

Lectures 13–14: Green’s functions (propagators) and $T$-products

- Inhomogeneous Klein-Gordon equation. Definition of Green function and formal solution with a Fourier transform. Poles in the integration path at $k_0 = \pm \omega(\vec{k})$. Possible deformations of the integration path: retarded, advanced and Feynman Green functions. Explicit expressions for the Feynman Green function: $\Delta_F(x) = \theta(x_0)\Delta^{(+)}(x) - \theta(-x_0)\Delta^{(-)}(x)$. Retarded Green function: $G_{\text{ret}}(x) = \theta(x_0)\Delta(x)$, with $\Delta = \Delta^{(+)} + \Delta^{(-)}$; advanced Green function: $G_{\text{adv}}(x) = \theta(-x_0)\Delta(x)$.
- Lorentz invariance of the Feynman propagator $i\Delta_F(x)$.
- Time-ordered products (in short: $T$-products):
  $\langle 0 | T(\phi(x)\phi^+(y)) | 0 \rangle = i\Delta_F(x-y)$. Equation for $\Delta_F(x)$: $(\Box + m^2)\Delta_F(x) = -\delta^{(4)}(x)$.
- Physical interpretation of the $T$-product: creation of a particle at $y$ and annihilation at $x$ if $x_0 > y_0$ or creation of an antiparticle at $x$ and annihilation at $y$ if $y_0 > x_0$.
- Consequences of microcausality: $[\phi(x), \phi^+(y)] = 0$ when $x$ are space like separated (i.e. causally disconnected), $(x-y)^2 < 0$. Calculation of the commutator $[\phi(x), \phi^+(y)] = i\Delta(x-y)$ for generic $x-y$.

Lectures 15–26: The Dirac equation

Classical theory

- Lecture 15. Mini-historical introduction to the Dirac equation. Emergence of the Klein-Gordon equation and problems in interpreting $\phi$ as a wavefunction (second-order time derivative equation; problems in defining a positive-definite probability density).
- The Dirac equation. Anticommutation relations of the $\alpha^i$ and $\beta$ matrices. Feynman form of the Dirac equation. The $\alpha, \beta$ and $\gamma$ matrices are traceless. Minimal dimension
of the Dirac spinors.

- Lecture 16-18, Spin of the Dirac field. Helicity (Homework).
- Covariance of the Dirac equation. Transformation properties of the Dirac spinors, \( \psi'(x') = S(\Lambda)\psi(x) \), with \( S(\Lambda) = \exp \left[ -(i/4)\sigma_{\mu\nu}A^{\mu\nu} \right] \) and \( \sigma^{\mu\nu} := (i/2)[\gamma^\mu, \gamma^\nu] \). Form of \( S(\Lambda) \) for rotations (Dirac and chiral representation) and boosts (in the chiral representation). Rotations by \( 2\pi \): \( \psi'(e^{2\pi i}x) = -\psi(x) \).
- Lecture 19-21, Massless limit of the Dirac equation: neutrinos and anti-neutrinos.
- From the Dirac equation to the Dirac action: \( \gamma^\mu \sigma^{\mu\nu} \gamma^0 = (\sigma^{\mu\nu})^\dagger \). Dirac adjoint spinor, \( \bar{\psi} := \psi^\dagger \gamma^0 \). Feynman’s slash notation \( \bar{p} := p_\mu \gamma^\mu \). The Dirac Lagrangian \( \mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \).
- Symmetries of the Dirac action. U(1) symmetry and its Noether current \( J^\mu = \bar{\psi}\gamma^\mu \psi \).
- Translational and Lorentz symmetry and the associated Noether currents. Energy momentum tensor \( T_{\mu\nu} = i\bar{\psi}\gamma^\mu \partial^\nu \psi - \mathcal{L} g_{\mu\nu} \) (associated to four-translations) and \( M_{\mu\nu}^\rho = x^\nu T_{\mu\rho} - x^\rho T_{\mu\nu} + \bar{\psi}\gamma^\mu \sigma_{\nu\rho} \psi / 2 \).
- Solving the Dirac equation: positive and energy solutions.
- Expansion of the free field:

\[
\psi(x) = \sum \frac{1}{(2\pi)^3} \int \frac{d^3p}{2\omega(p)} \left[ a_r(p)u_r(p)e^{-ipx} + b_r^\dagger(p)v_r(p)e^{ipx} \right].
\]

- The \( u_r(p) \) and \( v_r(p) \) spinors in the Dirac and chiral representation.
- Normalisations (as in Peskin-Schroeder): \( u_r^\dagger(p)u_s(p) = v_r^\dagger(p)v_s(p) = 2E(p) \delta_{rs} \), and \( \bar{u}_r(p)u_s(p) = -\bar{v}_r(p)v_s(p) = 2m \delta_{rs} \).
- Orthogonality relations:

\[
\bar{u}_r(p)v_s(p) = \bar{v}_r(p)u_s(p) = 0,
\]

- Completeness relations:

\[
\sum_r u_r(p)\bar{u}_r(p) = \bar{p} + m, \quad \sum_r v_r(p)\bar{v}_r(p) = \bar{p} - m, \quad \sum_r u_r(p)\bar{u}_r(p) - v_r(p)\bar{v}_r(p) = 2m.
\]

### Lectures 21–25: Quantisation of the Dirac field

- Expression for \( H \):

\[
H = \int \frac{d^3p}{(2\pi)^3} \omega(p) \sum_{r=1}^{2} \left[ a_r^\dagger(p)a_r(p) - b_r^\dagger(p)b_r(p) \right],
\]

and note the crucial minus sign.
- The need of anti-commutation relations for the \( a \) and \( b \) operators:

\[
\{a_r(\vec{k}), a_s^\dagger(\vec{k}')\} = \{b_r(\vec{k}), b_s^\dagger(\vec{k}')\} = (2\pi)^3 \delta_3(\vec{k} - \vec{k}').
\]
- Canonical anti-commutation relations:

\[
\{\psi(t,\vec{x}), \psi^\dagger(t',\vec{y})\} = \delta_{ij}\delta^{(3)}(\vec{x} - \vec{y}).
\]
- Second-quantised expressions for \( P^i := \int d^3x :T^{0i}: \) and \( Q := \int d^3x :\psi^\dagger \psi: \) in terms of the ladder operators \( a \) and \( b \).
- Total angular momentum for the Dirac field. Spin of the associated particles: \( J^3 a_r^\dagger(\vec{p} = \vec{0})|0\rangle = \frac{(-1)^{r+1}}{2} a_r^\dagger(\vec{p} = \vec{0})|0\rangle \), \( J^3 b_r^\dagger(\vec{p} = \vec{0})|0\rangle = \frac{(-1)^{r+1}}{2} b_r^\dagger(\vec{p} = \vec{0})|0\rangle \), where \( u_r(\vec{0}) = \sqrt{2m} \begin{pmatrix} \chi_r \\ 0 \end{pmatrix}, \quad v_r(\vec{0}) = \sqrt{2m} \begin{pmatrix} 0 \\ \eta_r \end{pmatrix} \) with \( \chi_1 = \eta_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \chi_2 = \eta_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \).
• The Dirac propagator. \( \langle 0 | T(\psi(x)\bar{\psi}(y))|0 \rangle = iS_F(x-y) \), where \( S_F(x) = (i\partial + m)\Delta_F(x) \) and for fermions the \( T \)-product is defined as \( T(\psi(x)\bar{\psi}(y)) := (\psi(x)\bar{\psi}(y))\theta(x_0-y_0) - \bar{\psi}(y)\psi(x)\theta(x_0-y_0) \).

• Equation for the Dirac propagator: \( (i\partial - m)(iS_F)(x) = i\delta^{(4)}(x) \). Fourier transform of the Dirac propagator, \( iS(p) = \frac{i}{p^2 - m^2 + i\epsilon} \).

Non-relativistic limit of the Dirac equation

• Lecture 25. Non-relativistic limit of the Dirac equation. “Small” and “large” components. Gyromagnetic ratio. An extraordinary success of Dirac’s equation: prediction of \( g = 2 \) from Dirac’s equation, explanation of the anomalous Zeeman effect.

Lecture 26: Discrete symmetries

• Parity. Transformation of spinors under parity: \( \psi'(x') = \gamma^0 \psi(x) \) where \( x' = (x^0, -x) \). Transformation properties bilinears under parity of Dirac: \( \bar{\psi}\psi \) (scalar), \( \bar{\psi}\gamma\mu\psi \) (vector), \( \bar{\psi}\gamma^5\psi \) (pseudo-scalar), \( \bar{\psi}\gamma\mu\gamma^5\psi \) (pseudo-vector).

• Charge conjugation. \( \psi_c = (C\gamma^0)\psi^* \) where \( C^{-1}\gamma^\mu C = -\gamma^\mu T \), and \( C = \gamma^2\gamma^0 \).

Lecture 27–33: Interactions

Time-evolution in the interaction picture

• Schrödinger, Heisenberg pictures. Decomposition of the Hamiltonian a free and an interacting part, \( H = H_0 + H_I \). The interaction picture and its connection to the Schrödinger and Heisenberg pictures.

• The \( S \)-matrix.

• Dyson’s expansion of the \( S \)-matrix:

\[
S = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \cdots dt_n \; T \left[ H_I(t_1) \cdots H_I(t_n) \right].
\]

In terms of the interaction Lagrangian: \( S = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n T \left[ \mathcal{L}_I(x_1) \cdots \mathcal{L}_I(x_n) \right]. \)

The emergence of \( T \)-products.


• Wick theorem (without general proof). Application to time-ordered products of normal products.

Feynman diagrams

• Simple four-point scattering processes in \( \phi^4 \) theory in four dimensions. Zeroth-order (disconnected) term; first-order term. Feynman rules for momentum conservation and vertices (and for external lines – trivial for scalar theories!)

• Lecture 31-33. Four-point scattering in \( \phi^3 \) theory in four dimensions: second-order term. Feynman rules for propagators and vertices. The total answer is the sum of the \( s \)-, \( t \)- and \( u \)-channel diagrams.
• A quick glance at loop diagrams: four-point scattering at one loop in $\phi^4$ theory.

• Superficial degree of divergence of a Feynman diagram, $D$. $D = 4 - E$ in $\phi^4$ theory in four dimensions, where $E$ is the number of external legs of the Feynman diagram.

• The interaction Lagrangian for Yukawa theory, $\mathcal{L}_Y = -g\bar{\psi}\psi\phi$. Electron-electron scattering in Yukawa theory to second order in the coupling constant.

$T$-products in perturbation theory$^{(*)}$

• $T$-products of elementary fields in the Heisenberg representation.

• A key result: $\langle 0|T(\phi_H(x_1)\cdots\phi_H(x_n))|0\rangle = \frac{\langle 0|T(\phi_I(x_1)\cdots\phi_I(x_n))\exp\left[-i\int d^4x\mathcal{H}_I\right]|0\rangle}{\langle 0|T\exp\left[-i\int d^4x\mathcal{H}_I\right]|0\rangle}$

where $\phi_H$ and $\phi_I$ are the fields in the Heisenberg and interaction picture, respectively.

• Examples of applications. Four-point function in $\phi^4$ theory. A first glance at LSZ reduction.

$^{(*)}$ Not required for the exam.