What is the functional form of the torque on the spin?

\[ \vec{\tau} = \vec{r} \times \vec{F} = |\vec{r}| |\vec{F}| \sin \theta \]

\[ \vec{r} \] - radius of body (moment from spin axis)

\[ F \] - tidal force

\[ \sim (mass \ of \ tidal \ bulges) \times tidal \ acceleration \]

or

\[ |\vec{\tau}_{\text{spin}}| = R_p M_{\text{bulge}} a_{\text{tidal}} \sin \gamma \]

Phase lag - angle between bulges and planet-satellite line
WHAT ARE THE MASSES OF THE BULGES?

- Assume each bulge has the same mass
- Assume they are each a hemisphere or shell with
  \[ \text{radius} = R_d \]
  \[ \text{thickness} = H \]

(Height of EQ TIDE)

See wk 4

Just these shells are
the 'bulges'

Mass of Bulge = \( \frac{(\text{volume of shell})}{\text{(density)}} \times \frac{\text{mass of bulge}}{\text{density}} = \frac{2\pi R^2 H \rho}{\rho} \)
RECALL THAT:

$$H_1 = \frac{5}{3} k_2 \left(\frac{M_2}{M_1}\right) \left(\frac{R_2}{r}\right)^3 R_1$$

HEIGHT OF THE EQUILIBRIUM TIDE

AND

$$k_2 = \frac{3/2}{1 + \frac{19\mu}{2\rho g R}}$$

IS THE LOVE NUMBER

(SEE DATA HEIGHTS SCANNED NOTES.)

ALSO NOTE THAT

$$\rho = \frac{M}{V} = \frac{3M}{4\pi R^3}$$
\[ M_{\text{bulge}} \approx (2\pi R^2) \left( \frac{5}{3} k_2 \left( \frac{M_2}{M_1} \right) \left( \frac{R_1}{r} \right)^3 \right) \left( \frac{3 M_1}{4\pi R_1^3} \right) \]
\[ \approx \frac{5}{2} k_2 M_2 \left( \frac{R_1}{r} \right)^3 \]

**Note**: \( \alpha \sim \frac{1}{r^3} \) dependence from \( H \)

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So the tidal torque is

\[ |\Gamma_{\text{spin}}| \propto R_0 \cdot M_{\text{bulge}} \cdot Q \cdot \sin S \]

\[ Q \propto \frac{2GM_2 R_1}{r^3} \]

\( Q \) near surface of \( R_1 \)

\[ \sin S \approx \frac{1}{Q} \]

\[ |\Gamma| \approx R_1 \left( \frac{5}{2} k_2 M_2 \left( \frac{R_1}{r} \right)^3 \right) \left( \frac{2GM_2 R_1}{r^3} \right) \frac{1}{Q} \]

\[ |\Gamma| \approx 5 \left( \frac{k_2}{Q} \right) \frac{GM_2^2}{r^6} R_1^5 \]
NOTE: THIS IS A HEURISTIC DERIVATION

• Assuming spherical shells for
  the bulge leads to errors
  of order factors of a few.

A rigorous derivation yields

\[ \Gamma_i^{\text{spin}} = \frac{3}{2} \left( \frac{K}{Q} \right) \frac{GM^2}{r^6} R_i^5 \]

for the magnitude of the torque.

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Note: For a closed system we have
\[
\frac{d\Gamma}{dt} = 0 = \Gamma_{\text{NET}}
\]

So

\[
\Gamma_{\text{SPIN}} + \Gamma_{\text{ORBIT}} = \Gamma_{\text{NET}}
\]

and

\[
0 = \Gamma_{\text{SPIN}} + \Gamma_{\text{ORBIT}}
\]

or

\[
\Gamma_{\text{ORBIT}} = -\Gamma_{\text{SPIN}}
\]
For the torque on the orbit we have

\[ \tau = \frac{3}{2} \left( \frac{K}{Q} \right) \frac{GM_2}{r^0} \frac{r^5}{r^0} \text{sign} (s_r - \omega_z) \]

\( \omega_1 \) - Rotational angular velocity of spinning body.

\( \omega_2 \) - Orbital angular velocity of orbiting body.

We can now use this to quantitatively evaluate how the angular momentum of the orbit and spin evolve.

We will now do examples with the Earth-Moon system.
A word on the Q of Planets and Satellites

- We generally don't know this from first principles.

- We calculate it from measuring \( \frac{da}{dt} \) or \( \frac{d\Omega}{dt} \) of planets/satellites.

- Or we estimate it from constraints on how much a system may evolve.

- Tidal evolution doesn't allow satellites to cross the synchronous orbit \( \Omega_s \).

\[ \text{where} \quad \Omega_s = \omega \]