SPA5201 - Physics Laboratory

Lecture 3

Errors and error propagation

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<u>Every</u> measured quantity has an associated <u>error</u>

It follows that any quantity calculated using measurements will also have an associated error

A physical quantity quoted without an associated error and relevant units is <u>meaningless</u>

Systematic versus statistical errors

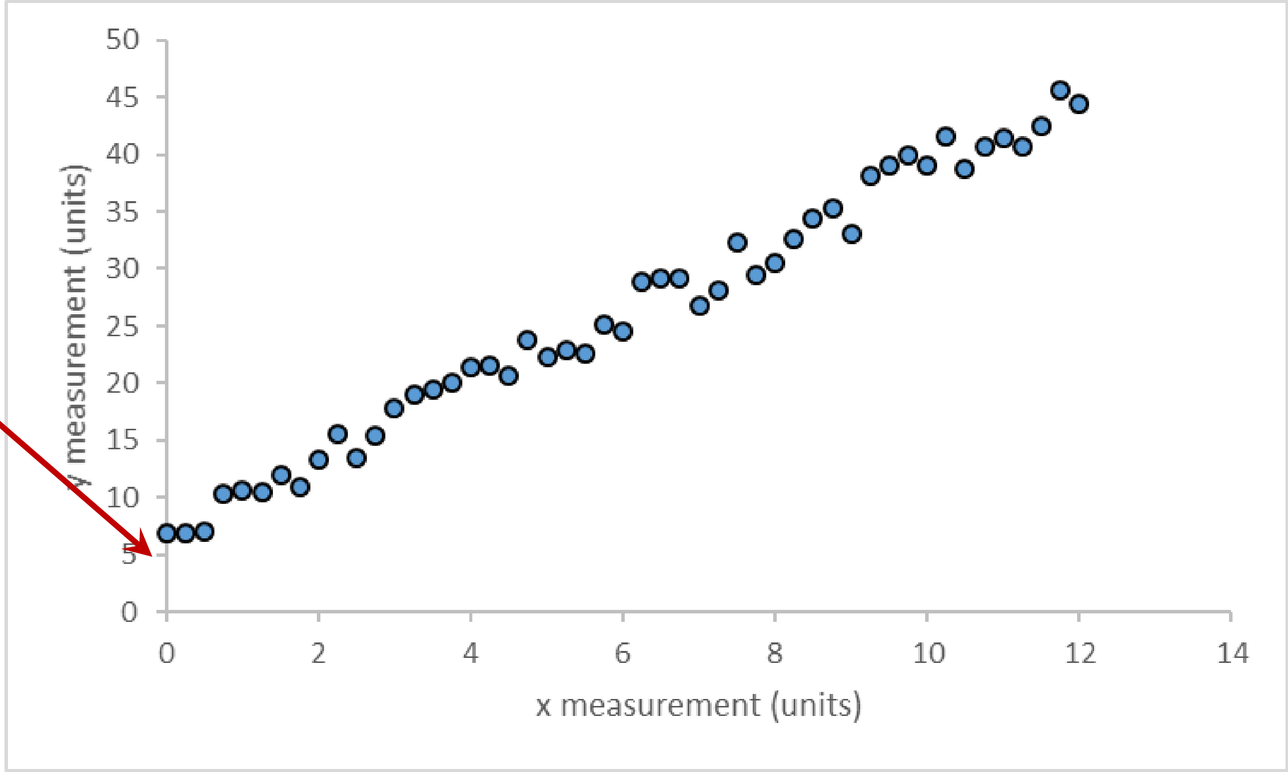
- The systematic error is constant over all measurements (so the error averaged over all measurements is not expected to be zero)
  - Commonly appear as the addition (or subtraction) of a constant to all measurements, e.g. a ruler with the first 4 mm shaved off will always read 4 mm “over” the “true” reading
  - Can sometimes appear as a multiplicative factor affecting all measurements (e.g. an initial calibration error)
  - Commonly result from a defective instrument or procedure
- The statistical (random) error is different for each measurement (therefore the average error over all measurements is expected to be zero for a random error)

Systematic errors *can* be eliminated graphically

- Suppose we have a theoretical relationship: $y = mx$

…and we plot experimental y versus experimental x

The non zero intercept could be due to a systematic error in <u>either</u> the x or y measurements



Caveat!

- A systematic error will affect all points (up, down, right, left) equally (so linear plots are no problem)
- Other plots have to be chosen with care
  - In a logarithmic plot this will <u>not</u> appear as a simple shift since

$$\ln(x + C) \neq \ln(x) + \ln(C)$$

…but a calibration error would (a constant multiplicative factor)

$$\ln(Cx) = \ln(x) + \ln(C)$$

random errors

- Can arise from the measurement method
  - e.g. d = 5 mm, 5 mm, 5 mm, 5 mm using a mm scale
  - instrumental error of ± ½ a division (granularity, since you would either “round up” or “round down”)
  - d = 5 ± 0.5 mm
  - Often overlooked in digital readouts!

- Can arise from actual variations in a reading or readings
  - e.g. d = 5.1 mm, 4.9 mm, 5.2 mm, 4.9 mm, 5.1 mm, 4.8 mm
  - d = 5.0 ± 0.15 mm

- Have a mean value
- How do we evaluate the error?

random errors

- Can use half the range (largest – smallest)
  - But improbable readings can “throw” the estimate
  - e.g. I = 5.1 mA, 4.9 mA, 5.2 mA, 4.9 mA, 6.0 mA, 4.8 mA, 5.0 mA
  - Can be useful for fluctuating readings

- Best to use the standard deviation

Mean and standard deviation

- Mean, $\langle x \rangle$ or $\bar{x}$

$$\langle x \rangle = \frac{1}{N} \sum_{n=1}^{n=N} x_n$$

- Standard deviation of x, σ_x

$$\sigma_x = \sqrt{\frac{1}{N-1} \sum_{n=1}^{n=N} (x_n - \bar{x})^2}$$

errors to be indicated for <u>all</u> quantities

- you should indicate how you arrived at your estimate
  - Instrumental (measurement) if a single unvarying reading
  - Half range if the reading is fluctuating
  - Standard deviation if you have repeated readings

- Please consider the repeatability and reproducibility of a reading
  - Repeated readings (e.g. same instrument and/or sample)
  - Reproduced readings (e.g. different instruments and/or samples)

Quote numerical values to the correct number of significant figures (dictated by the error)

- Rules of thumb
  - Look at the error,
  - if the first significant figure in the error < 4 use two significant figures in the error
  - if it is > 4 use one significant figure in the error

- Keep the precision the same for the mean and the error
- e.g.

5.3293 ± 0.1572 units should be quoted as 5.33 ± 0.16 units

238.567 ± 52.15 units should be quoted as 240 ± 50 units

the error in the mean

- We can define $\bar{x}$, σ_x
- however, an increased number of readings (N) will improve our confidence in the mean value

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}$$

…reduces the error in the mean

(often quoted as a "standard error" in software packages)

Quoting errors

So we have a set of experimentally obtained quantities:

$\bar{x} \pm \sigma_x$, $\bar{y} \pm \sigma_y$, $\bar{z} \pm \sigma_z$, $\bar{q} \pm \sigma_q$ etc.

…we can quote absolute errors

σ_x units, σ_y units, σ_z units, σ_q units

…or relative errors

$$\frac{\sigma_x}{\bar{x}}, \frac{\sigma_y}{\bar{y}}, \frac{\sigma_z}{\bar{z}}, \frac{\sigma_q}{\bar{q}}$$

<u>all dimensionless</u>

error propagation

We use

$\bar{x} \pm \sigma_x$, $\bar{y} \pm \sigma_y$, $\bar{z} \pm \sigma_z$, $\bar{q} \pm \sigma_q$ etc.

to calculate a further quantity, $\mathrm{C} = F(x, y, z, q)$

what is the error in the calculated quantity, $\mathrm{C} = F(x, y, z, q)$?

i.e. how do we obtain $C \pm \sigma_C$ units?

error propagation

obtaining $\mathrm{C} = F(x, y, z, q)$ is trivial…

…just substitute $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{q}$ into $F(x, y, z, q)$

what about σ_C ?

a general result:

$$\sigma_c^2 = \left(\frac{\partial F}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial F}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial F}{\partial z}\right)^2 \sigma_z^2 + \left(\frac{\partial F}{\partial q}\right)^2 \sigma_q^2 + \cdots$$

Why the quadrature?

- Because we are assuming the errors in the different variables to be independent…

… x could be “high” while y is “low” etc.

so we do not overestimate the error

simple (memorable) rules

- For addition or subtraction add the absolute errors in quadrature

  e.g. if $C = x \pm y$, then $\sigma_c^2 = \sigma_x^2 + \sigma_y^2$

- For multiplication and division add fractional errors in quadrature

  e.g. if $C = \frac{xz}{y}$, then $\left(\frac{\sigma_C}{\bar{C}}\right)^2 = \left(\frac{\sigma_x}{\bar{x}}\right)^2 + \left(\frac{\sigma_y}{\bar{y}}\right)^2 + \left(\frac{\sigma_z}{\bar{z}}\right)^2$

- For powers the fractional error is scaled by the power

  e.g. if $C = x^n$, then $\left(\frac{\sigma_C}{\bar{C}}\right)^2 = n^2 \left(\frac{\sigma_x}{\bar{x}}\right)^2$ i.e $\frac{\sigma_C}{\bar{C}} = n \frac{\sigma_x}{\bar{x}}$

- Remember to convert back into an absolute error from the relative error in your final result!

$$\text{i.e. } \sigma_C = \left(\frac{\sigma_c}{\bar{C}}\right)\bar{C}$$

Comparing values

- Suppose you wish to compare two values for the same quantity…
- $Q_1 \pm \sigma_{Q_1}$ and $Q_2 \pm \sigma_{Q_2}$
- e.g. 5.5 ± 0.5 eV and 6.28 ± 0.01 eV

…one could be a literature value, for example…

- How do we assess the level of agreement or disagreement?
- How do we compare as Physicists?

Comparing values

- calculate the difference in the values

$$\Delta Q = |Q_1 - Q_2|$$

- Calculate the error in the difference

$$(\sigma_{\Delta Q})^2 = (\sigma_{Q_1})^2 + (\sigma_{Q_2})^2$$

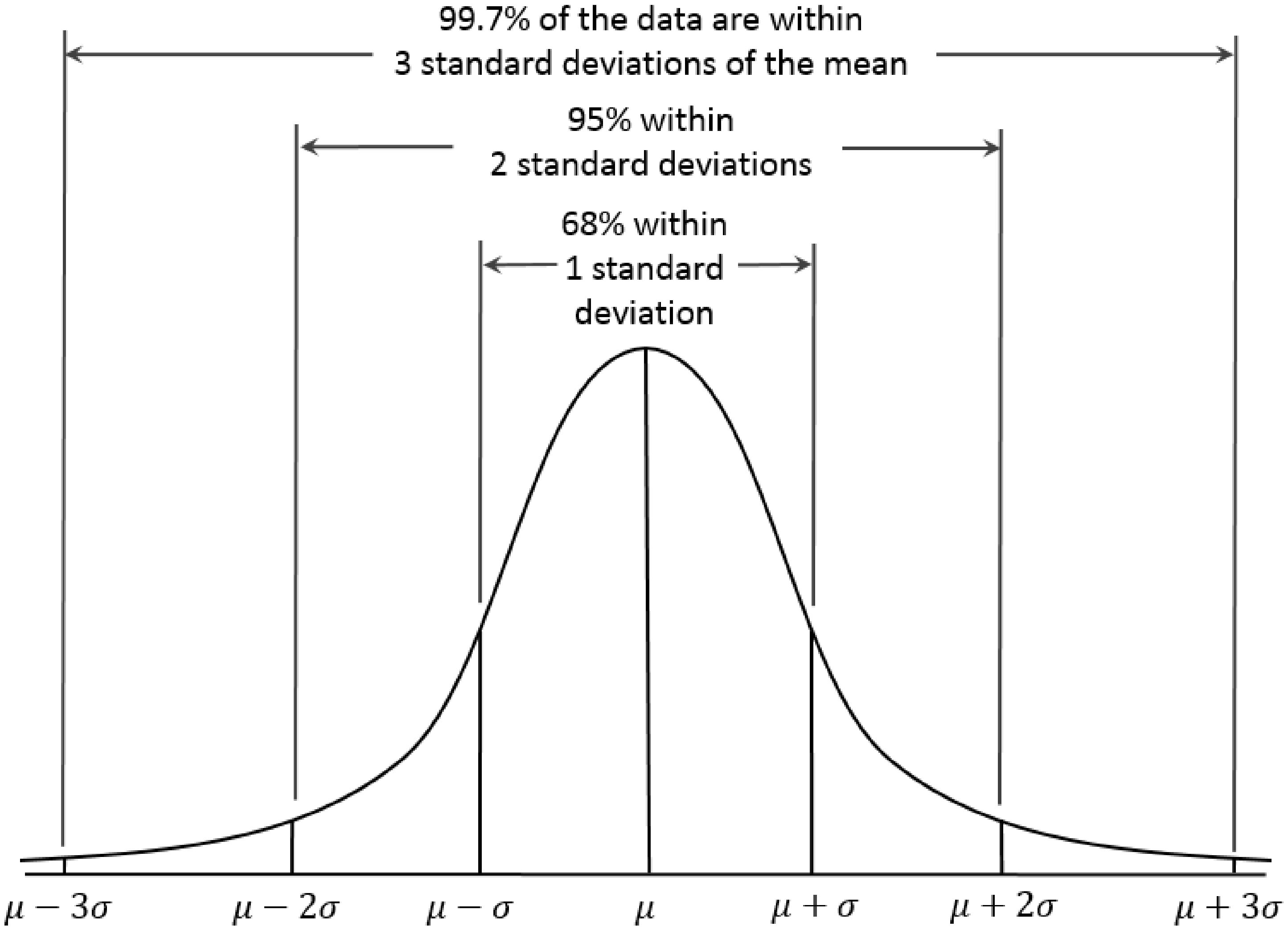
- Then express ΔQ in terms of $\sigma_{\Delta Q}$ i.e. evaluate

$$\frac{\Delta Q}{\sigma_{\Delta Q}}$$

- Generally for $\frac{\Delta Q}{\sigma_{\Delta Q}} < 3$ the values are deemed to agree within error (3 sigma test)

…else they disagree

How many sigmas do we need?



within 4σ: 0.99994
within 5σ: 0.9999997

The error analysis and propagation outlined so far applies to all experimental values, including values obtained from curve fitting.

… see lecture 4 (Friday)

epilogue

recall comparing 5.5 ± 0.5 eV and 6.28 ± 0.01 eV

ΔE = 0.78 eV

$\sigma_{\Delta E}$ = 0.5 eV (well 0.5000999, meh…)

$\frac{\Delta E}{\sigma_{\Delta E}}$ = 1.56

…we agree (sort of!)