Queen Mary University of London

MSc/MSci Examination

Main Examination Period 2019

SPA7010P / SPA7010U The Galaxy Duration: 2 hours 30 minutes

YOU ARE NOT PERMITTED TO READ THE CONTENTS OF THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR.

Instructions:

Answer ALL questions from Section A. Answer ONLY TWO questions from Section B. Section A carries 50 marks, each question in section B carries 25 marks.

If you answer more questions than specified, only the first answers (up to the specified number) will be marked. Cross out any answers that you do not wish to be marked.

Only non-programmable calculators are permitted in this examination. Please state on your answer book the name and type of machine used.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

Important note: The academic regulations state that possession of unauthorised material at any time when a student is under examination conditions is an assessment offence and can lead to expulsion from QMUL.

Please check now to ensure you do not have any notes, mobile phones or unauthorised electronic devices on your person. If you have any, raise your hand and give them to an invigilator immediately. It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption is also an assessment offence.

EXAM PAPERS MUST NOT BE REMOVED FROM THE EXAM ROOM.

Examiners:
Dr. N. Cooper
Prof. R. Nelson

SECTION A Answer ALL questions in Section A

Question A1

Sketch Hubble's "tuning fork diagram" which illustrates the morphological classification of normal galaxies, and explain how galaxy properties (colours, spectra, and gas and dust content) vary across the diagram.

[6 marks]

Question A2

Explain the difference between collisional and collisionless processes in galaxy formation. Explain why gas clouds in a galaxy tend to settle into a rotating disk.

[4 marks]

Question A3

Explain the difference between "pressure support" and "rotational support" for a galaxy. Which of these dominates for spiral and elliptical galaxies respectively?

[4 marks]

Question A4

State (without proof) the virial theorem for a self-gravitating stellar system, defining the terms used. What are the conditions required for its application?

A cluster of galaxies has an observed radius of $800\,\mathrm{kpc}$ and a velocity dispersion of $800\,\mathrm{km\,s^{-1}}$. Use the virial theorem to estimate the total mass of the cluster, giving your answer in solar masses.

Hint: you may assume the potential energy of a uniform sphere of mass M and radius R is $U = -3GM^2/(5\,R)$, where G is the gravitational constant.

[6 marks]

Question A5

Define the term "integral of motion" for a function of position and velocity in a galaxy. State (without proof) two integrals of motion for a galaxy with a time-independent and axisymmetric potential.

[4 marks]

Question A6

Explain the meaning of the terms HI, HII and H_2 referring to hydrogen in the Galaxy, and describe the typical environments in which they are located.

[6 marks]

Question A7

Give the standard definitions of the symbols X, Y, Z, used in the study of Galactic chemical evolution. Quote typical values for these in the local interstellar medium. State which processes respectively produced (a) most of the helium, and (b) most of the heavy elements.

[4 marks]

Question A8

A star near the Galactic plane is observed to have apparent magnitudes in the blue and visual bands of $B = 12.49$, $V = 11.79$. Comparison of the star's spectrum with standard spectra indicates that it has absolute magnitudes $M_B = 3.20$, $M_V = 2.70$ in these bands. Assuming that the reddening ratio for interstellar dust is $A_V/E(B-V) = 3.0$, estimate (i) the reddening of the star, (ii) the V-band extinction in magnitudes, and (iii) the distance to the star.

[5 marks]

Question A9

For a gravitational lens with perfect alignment, the angular Einstein ring radius, in radians, is given by

$$\theta_E = \sqrt{\frac{4GM}{c^2}\frac{D_{LS}}{D_S D_L}}\,,$$

where M is the lens mass, D_L is the distance from Earth to the lens, D_S is the distance from Earth to the source, D_{LS} is the distance between lens and source, G is the gravitational constant and c is the speed of light.

Assuming a source star in the Large Magellanic Cloud at $D_S = 50\,\mathrm{kpc}$, and a lens of mass $0.1\,M_\odot$ at distance $10\,\mathrm{kpc}$, evaluate θ_E in arcseconds.

Comment on any implication for possible observations of microlensing phenomena.

[5 marks]

Question A10

Briefly describe ESA's GAIA mission, including its mode of operation, main goals, and how it will improve on the earlier Hipparcos survey in terms of accuracy and data volume. Briefly mention one recent published scientific highlight based on GAIA data.

[6 marks]

SECTION B Answer TWO questions from Section B

Question B1

a) Define the terms *weak encounter* and *strong encounter* for two stars approaching each other in a large stellar system.

[2 marks]

b) In a weak encounter between two stars each of mass m with relative velocity v, the change in the velocity of one star in the reference frame of the other is given by

$$\delta v = \frac{2Gm}{bv},$$

where G is the constant of gravitation and b is the impact parameter.

A star moves through a spherical region of overall radius R containing N stars distributed uniformly in space. If the mean change in the square of the velocity is $\delta(v^2) = (\delta v)^2$ in a single weak encounter, show that the changes in v^2 caused by weak encounters with impact parameters in the range b to $b + \mathrm{d}b$ during a time t is

$$\Delta v^2 = \left(\frac{2Gm}{bv}\right)^2 \left(\frac{3bvtN\mathrm{d}b}{2R^3}\right),$$

and hence show that the total change in v^2 in a time t caused by weak encounters with all impact parameters is

$$\Delta v^2(t) = 6\left(\frac{Gm}{v}\right)^2 \frac{v\,t\,N}{R^3} \ln\left(\frac{b_{\text{max}}}{b_{\text{min}}}\right),$$

where b_{max} and b_{min} are the largest and smallest values of the impact parameter.

[7 marks]

c) From the result above, write down an expression for the relaxation time T_{relax}; hence show that for suitable choices of b_{min} and b_{max}, the ratio of the relaxation time to the crossing time, T_{cross}, is given approximately by

$$\frac{T_{\text{relax}}}{T_{\text{cross}}} \approx \frac{N}{12 \ln N}.$$

Hint: you may assume that in a stellar system of radius R containing N stars each of mass m, the typical velocity v is given by $v \approx \sqrt{GNm/R}$.

[7 marks]

d) The dark matter halo of our Galaxy has been modelled as a spherical halo with density profile

$$\rho(r) = \frac{\rho_0\, a^2}{r^2 + a^2} \ ,$$

where r is the Galactocentric radius, ρ_0 is the halo central density and $a = 5\,\text{kpc}$ is the core radius. Observations indicate that the circular velocity of the Sun around the Galaxy is approximately $220\,\text{km}\,\text{s}^{-1}$, and the dark halo contributes approximately half of the total mass inside the Sun's orbit.
Hence estimate the local density of dark matter ρ at $r = R_0 = 8\,\text{kpc}$, in units of $M_\odot\,\text{pc}^{-3}$.

Hint: you may quote the standard integral $\displaystyle\int \frac{x^2}{x^2 + a^2}\, dx = x - a \tan^{-1}\frac{x}{a} + \text{const}$.

[9 marks]

Question B2

a) The collisionless Boltzmann equation states that

$$\frac{\partial f}{\partial t} + \sum_{j=1}^{3} \left(\frac{dx_j}{dt} \frac{\partial f}{\partial x_j} + \frac{dv_j}{dt} \frac{\partial f}{\partial v_j} \right) = 0 ,$$

where $f(\mathbf{x}, \mathbf{v}, t)$ is the distribution function, x_j and v_j are the j-components of position and velocity, and t is time.

Derive from this the second Jeans equation,

$$\frac{\partial (n\langle v_i \rangle)}{\partial t} + \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \Big(n\langle v_i v_j \rangle \Big) = -\frac{\partial \Phi}{\partial x_i} n ,$$

where n is the number density of stars, $\langle v_i \rangle$ is the mean value of the v_i velocity component at a point, and $\Phi(\mathbf{x}, t)$ is the gravitational potential.

Explain your definitions, working and assumptions throughout.

[10 marks]

b) One of the Jeans equations in a cylindrical coordinate system (R, θ, z) with the origin at the centre of the Galaxy, and $z = 0$ in the Galactic plane, can be expressed as

$$\frac{\partial (n\langle v_z \rangle)}{\partial t} + \frac{\partial (n\langle v_R v_z \rangle)}{\partial R} + \frac{\partial (n\langle v_z^2 \rangle)}{\partial z} + \frac{n\langle v_R v_z \rangle}{R} = -n \frac{\partial \Phi}{\partial z} ,$$

where n is the star number density, v_R and v_z are the velocity components in the R and z directions, $\Phi(R, z)$ is the Galactic gravitational potential and t is time. Assuming that the Galaxy is in a steady state, show that the surface mass density $\Sigma(z, R_0)$ within a distance z of the mid-plane of the Galactic disc at the solar radius R_0 in the direction of the Galactic poles is given by

$$\Sigma(z, R_0) \simeq \frac{-1}{2\pi G n} \frac{\partial}{\partial z} \left(n\langle v_z^2 \rangle \right) .$$

Explain the assumptions you made.

[8 marks]

c) A spherically-symmetric galaxy is dark-matter dominated and has a gravitational potential

$$\Phi(r) = -\frac{GM_{\text{tot}}}{r + a}$$

at a radial distance r from its centre, where a is a positive constant and M_{tot} is the total mass.

A population of stars is distributed within this potential, and the stars contribute negligibly to the total density. The system of stars has an isotropic velocity distribution with a velocity dispersion σ that is constant across the galaxy, has zero net rotation, and has number density n_0 at the centre. Assuming that the potential is constant over time, derive an expression for the number density $n(r)$ of stars as a function of radius r.

Hint: you may quote a suitable Jeans equation from the Appendix below.

[7 marks]

Question B3

a) A galaxy is modelled using a spherically-symmetric gravitational potential of the form

$$\Phi(r) \;=\; -\,\frac{4\pi G k}{r}\,\ln\left(\frac{r+a}{a}\right)\;,$$

where r is the radial distance from the centre of the galaxy, a and k are constants and G is the constant of gravitation. Using Poisson's equation $\nabla^2\Phi = 4\pi G\rho$, show that the mass density ρ as a function of distance r implied by this potential is

$$\rho(r) \;=\; \frac{k}{r(r+a)^2}\;.$$

Hint: you may use a suitable equation from the Appendix below.

[8 marks]

b) The distance l from the Milky Way to M31 (the Andromeda galaxy) is assumed to satisfy the differential equation

$$\frac{d^2 l}{dt^2} = -\frac{GM}{l^2}$$

with the constant M being the total mass of the Local Group. Verify that

$$t \;=\; \tau_0(\eta - \sin\eta)\;,$$
$$l \;=\; \left(GM\tau_0^2\right)^{\frac{1}{3}}(1-\cos\eta)$$

with τ_0 a constant, and η a parameter, is a solution of the above differential equation.

[6 marks]

c) Explain how observable quantities in the above equations may be used to estimate the total mass M of the Local Group.

[3 marks]

d) For the potential in part (a) above, find an expression for the circular velocity $v_c(r)$; and hence show that the logarithmic derivative of this is given by

$$\frac{d\ln v_c}{d\ln r} \equiv \frac{r}{v_c}\,\frac{dv_c}{dr} = \frac{\dfrac{r+a}{r}\ln\left(\dfrac{r+a}{a}\right) - \dfrac{r}{r+a} - 1}{2\left[1 - \dfrac{r+a}{r}\ln\left(\dfrac{r+a}{a}\right)\right]}\;.$$

By evaluating the latter at $r = a$ and $r = 5a$ respectively, show that the rotation curve is approximately flat between these radii.

[8 marks]

Question B4

a) General Relativity predicts that the bending angle α (assumed small) for a light ray passing a distance b from a compact object of mass M is given by

$$\alpha = \frac{4GM}{c^2 b} \ .$$

Show that for a lens system where the Earth, lens, and source are exactly collinear, the angular Einstein radius of a gravitational lens is

$$\theta_E \ = \ \sqrt{\frac{4GM_L}{c^2}\frac{D_{LS}}{D_L D_S}} \quad ,$$

where M_L is the mass of the lens, and D_S, D_L and D_{LS} are the distances from the observer to the light source, from the observer to the lensing object, and between the lens and source respectively. G is the universal gravitational constant and c is the speed of light.

[5 marks]

b) Sketch a light curve (flux vs time) for a typical gravitational microlensing event.

[3 marks]

c) The microlensing optical depth may be defined as the mean number of lenses present within the Einstein angle, θ_E.

Using the above expression for θ_E, show that the optical depth through a distribution of microlenses of mass M_L along the line of sight to a source is given by

$$\tau \ = \ \frac{4\pi G}{c^2 D_S} \int_0^{D_S} D_L \, D_{LS} \, \rho(D_L) \, \mathrm{d}D_L \quad ,$$

where $\rho(D_L)$ is the mean mass density of lenses in a volume of space at distance D_L.

[7 marks]

d) A survey attempts to detect microlensing events from MACHOs by observing a field in the Galactic Bulge close to the Galactic Centre. Assume that the dark matter halo is made from compact objects (MACHOs) with approximately stellar mass and with a density distribution

$$\rho(r) \ = \ \frac{\rho_0 \, a^2}{r^2 + a^2} \quad ,$$

where r is the radial distance from the Galactic Centre, ρ_0 is the central dark matter density and a is a constant. Show therefore that the optical depth of microlensing to the field is

$$\tau \ = \ \frac{2\pi G \rho_0 a^2}{c^2} \left(\ln\left(1 + \frac{R_0^2}{a^2}\right) \ + \ \frac{2a}{R_0} \tan^{-1} \frac{R_0}{a} \ - \ 2 \right) \ ,$$

where R_0 is the distance of the Sun from the Galactic Centre. You may assume that the star field is not significantly affected by dust extinction for this calculation. You may find helpful the standard integral

$$\int \frac{x\,(b-x)}{(b-x)^2+a^2}\ \mathrm{d}x \ = \ -\,x\ -a\tan^{-1}\left(\frac{b-x}{a}\right)\ -\ \frac{1}{2}b\ln\left(a^2+(b-x)^2\right)\ +\ \text{constant}\,.$$

Estimate τ to within an order of magnitude, given $R_0 = 8.0$ kpc, $a = 2.0$ kpc and $\rho_0 = 2.0 \times 10^{-20}$ kg m^{-3}. What does this imply for the number of stars that would have to be studied in the microlensing survey?

[10 marks]

End of Paper - An Appendix of 1 page follows

Appendix - Useful Information

In this paper, π and e represent the standard mathematical constants.
G is the gravitational constant, with $G = 6.67 \times 10^{-11}$ m^3 kg^{-1} s^{-2}.
c is the speed of light, with $c = 3.00 \times 10^8$ m s^{-1}.
1 parsec (pc) $= 3.09 \times 10^{16}$ m.
1 astronomical unit (AU) $= 1.50 \times 10^{11}$ m.
The mass of the Sun is $M_\odot = 2.0 \times 10^{30}$ kg.
The distance of the Sun from the Galactic Centre is $R_0 = 8.0$ kpc.

Poisson's equation states that $\nabla^2\Phi = 4\pi G\rho$ at any point in a gravitational field, where Φ is the gravitational potential, G is the constant of gravitation, and ρ is the mass density at that point.

The Laplacian of a scalar function Φ in a spherical coordinate system (r, θ, ϕ) is

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} .$$

The Laplacian of a scalar function Φ in a cylindrical coordinate system (R, ϕ, z) is

$$\nabla^2\Phi = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Phi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} .$$

The Jeans equations in a steady-state (i.e. time-independent), spherically-symmetric galaxy give the following result

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(n\,\langle v_r^2\rangle\right) + \frac{n}{r}\left[2\langle v_r^2\rangle - \langle v_\theta^2\rangle - \langle v_\phi^2\rangle\right] = -\,n\,\frac{\mathrm{d}\Phi}{\mathrm{d}r} ,$$

in spherical coordinates, where n is the number density of stars at a distance r from the centre, v_r, v_θ and v_ϕ are the components of the velocity in the r, θ and ϕ directions, and $\Phi(r)$ is the gravitational potential.

In the absence of cosmological effects, the apparent magnitude m of an astronomical object in a photometric band is related to its absolute magnitude M in that band and its distance D from the observer by

$$m - M = 5\log_{10}(D/\mathrm{pc}) - 5 + A ,$$

where A is the extinction in the same band expressed in magnitudes.

End of Appendix