SPA5302 — *NUCLEAR PHYSICS & ASTROPHYSICS*

EXERCISE SHEET 7 — 2019

1. Derive the Gamow factor for quantum tunnelling for a fusion reaction given in lectures.

> **Solution:** Use the derivation for alpha-decay, keeping the first term in eq 73 only.

2. Show that matter radiation equality in the early universe happens at temperature

$$T_{eq} = \frac{T_0}{a_{eq}} \approx 6.5 \times 10^4 \Omega_m h^2 \,\mathrm{K}\,,$$

> **Solution:**
>
> Lambda term can be ignored at this time. Equality given when the energy density of matter and radiation are equal, so $\Omega_m a^{-3} = \Omega_r a^{-4}$, where $H^2 = 2H_0^2 \Omega_r a_{eq}^{-4}$. We therefore have $a_{eq} = \Omega_r/\Omega_m$. The temperature scales as $T = T_0/a$, so $T_{eq} = T_0 \Omega_m/\Omega_r$. Putting in the numbers gives the result.

3. Show that in the far future the expansion rate will be constant and that the scale factor will grow exponentially.

> **Solution:** As $a \to \infty$, the Friedmann equation becomes
>
> $$H^2 = H_0^2 \Omega_\Lambda$$
>
> Since $H = \dot{a}/a$, integrating gives $a \sim \exp(H_0 \sqrt{\Omega_\Lambda} t)$

4. Argue that at very early times the cosmological scale factor grows as the square root of the time. Hence show

$$T(t) \sim 10^{10}\,\mathrm{K} \left(\frac{1\,\mathrm{s}}{t}\right)^{1/2} .$$

> **Solution:** At very early times we can ignore matter and dark energy as the radiation is dominant. Using the definition of the Hubble rate as $H = \dot{a}/a$ we have
>
> $$H = H_0 \sqrt{\Omega_r}/a^2 = \dot{a}/a$$

integrating this differential equation from $t = 0,\ a = 0$ gives

$$\frac{1}{2}a^2 = H_0\sqrt{\Omega_r}t \Rightarrow a = (2H_0\sqrt{\Omega_r}t)^{1/2}$$

Temperature follows from $a = T_0/T$:

$$T(t) = \frac{T_0}{(2H_0\sqrt{\Omega_r}t)^{1/2}} = \frac{2.725\,K}{(2\sqrt{4.2\times10^{-5}h^{-2}}t)^{1/2}}\sqrt{\frac{3.086\times10^{17}}{h}\,\mathrm{s}}$$

5. For $t \ll 1\,\mathrm{s}$ write down the weak interactions which keep the protons and neutrons in equilibrium.

Solution: see notes, p 88

6. If neutron freeze-out happens at $kT \sim 0.8\,\mathrm{MeV}$, what is the neutron fraction? If the mass of the neutron were much closer to the mass of the proton, would this fraction be more or less?

Solution: neutron fraction is $1/6$ - see notes p87. If $(m_n - m_p)c^2 \ll 1.3\,\mathrm{MeV}$ then

$$\frac{n_n}{n_p} \sim e^{-(m_n-m_p)c^2/0.8MeV}$$

this fraction would be much closer to 1, implying that the neutron fraction would be much closer to 50%

7. Show that for the formation of deuterium, the temperature T given by the solution to

$$\left(\frac{kT}{m_pc^2}\right)^{3/2} e^{B_D/kT} \sim \frac{1}{\eta},$$

must be reached. The solution to this is $kT \sim 0.06\mathrm{MeV}$ for $\eta \sim 6\times10^{-10}$. If the baryon to photon ratio was much larger, would you expect this to be much higher or lower? Would there be more helium or less?

Solution: We start using the number density equilibrium formula, for the ratio

$$n_d/n_nn_p \sim (m_d/m_nm_pc^2 1/kT)^{3/2}e^{-(m_d-m_n-m_p)c^2/kT}$$

which gives the exponential as the binding energy of d B_D. Now use $m_d \sim m_n$, and $n_n \sim n_{baryons} = \eta n_\gamma \sim \eta T^3$ to give n_d/n_p in terms of the baryon-photon ratio η:

$$n_d/n_p \sim \eta\left(\frac{kT}{m_pc^2}\right)^{3/2} e^{B_D/kT}$$

$n_d/n_p \sim 1$ gives the desired result.

T would be higher. Less n would have decayed meaning more He.

Some (potentially) useful information:

The radius of a nuclei may be approximated by $R \approx 1.2A^{1/3}$ fm.
The semi-empirical mass formula (SEMF) for the binding energy of a nucleon is

$$B(Z, A) = a_V A - a_S A^{2/3} - a_C \frac{Z(Z-1)}{A^{1/3}} - a_A \frac{(A-2Z)^2}{A} + \delta(Z, A)\,.$$

Constants in the SEMF: $a_V = 15.56, a_S = 17.23, a_C = 0.697, a_A = 23.28, a_P = 12.0$ where each number is in MeV.

Nuclear Shells: Protons

$$1s_{\frac{1}{2}} \Big\downarrow_2 1p_{\frac{3}{2}} \mid 1p_{\frac{1}{2}} \Big\downarrow_8 1d_{\frac{5}{2}} \mid 2s_{\frac{1}{2}} \mid 1d_{\frac{3}{2}} \Big\downarrow_{20} 1f_{\frac{7}{2}} \Big\downarrow_{28} 2p_{\frac{3}{2}} \mid 1f_{\frac{5}{2}} \mid 2p_{\frac{1}{2}} \mid 1g_{\frac{9}{2}} \Big\downarrow_{50} 1g_{\frac{7}{2}} \mid 2d_{\frac{5}{2}} \mid 1h_{\frac{11}{2}} \mid 2d_{\frac{3}{2}} \mid 3s_{\frac{1}{2}} \Big\downarrow_{82} 1h_{\frac{9}{2}} \mid 2f_{\frac{7}{2}} \mid \cdots$$

Shells: Neutrons

$$1s_{\frac{1}{2}} \Big\downarrow_2 1p_{\frac{3}{2}} \mid 1p_{\frac{1}{2}} \Big\downarrow_8 1d_{\frac{5}{2}} \mid 2s_{\frac{1}{2}} \mid 1d_{\frac{3}{2}} \Big\downarrow_{20} 1f_{\frac{7}{2}} \Big\downarrow_{28} 2p_{\frac{3}{2}} \mid 1f_{\frac{5}{2}} \mid 2p_{\frac{1}{2}} \mid 1g_{\frac{9}{2}} \Big\downarrow_{50} 2d_{\frac{5}{2}} \mid 1g_{\frac{7}{2}} \mid 1h_{\frac{11}{2}} \mid 2d_{\frac{3}{2}} \mid 3s_{\frac{1}{2}} \Big\downarrow_{82} 2f_{\frac{7}{2}} \mid 1h_{\frac{9}{2}} \mid \cdots$$

$\frac{e^2}{4\pi\varepsilon_0}$	$= 1.439965$ MeV fm
Boltzmann's constant	$k_B = 8.6173303 \times 10^{-5}$ eV/K
Planck's constant	$h = 4.135668 \times 10^{-15}$ eV s
Speed of light	$c = 2.99792 \times 10^8$ m/s
Neutron mean lifetime	881 s
Atomic mass unit	$1\,u = 931.4940954\,\text{MeV}/c^2 = 1.66054 \times 10^{-27}$ kg
Mass of electron	$m_e = 5.4858 \times 10^{-4}\,u = 0.51099895\,\text{MeV}/c^2$
Mass of proton	$m_p = 1.00727646688\,u = 938.27208\,\text{MeV}/c^2$
Mass of neutron	$m_n = 1.00866491578\,u = 939.56541\,\text{MeV}/c^2$
Mass of $^1_1$H	$= 1.00782503\,u$
Mass of $^2_1$H	$= 2.01410178\,u$
Mass of $^3_1$H	$= 3.01604927\,u$
Mass of $^3_2$He	$= 3.01602932\,u$
Mass of $^4_2$He	$= 4.00260325\,u$
Mass of $^{232}_{90}$Th	$= 232.038055\,u$
Mass of $^{234}_{90}$Th	$= 234.043601\,u$
Mass of $^{235}_{92}$U	$= 235.043930\,u$
Mass of $^{236}_{92}$U	$= 236.045568\,u$
Mass of $^{238}_{92}$U	$= 238.050788\,u$
Mass of $^{239}_{92}$U	$= 239.054293\,u$
Mass of $^{240}_{94}$Pu	$= 240.053811\,u$
Mass of $^{241}_{94}$Pu	$= 241.056849\,u$
Mass of $^{242}_{94}$Pu	$= 242.058741\,u$
Mass of the Sun	$M_\odot = 1.988 \times 10^{30}$ kg
Gravitational constant	$G = 6.67408 \times 10^{-11}\,\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}$

Nuclei masses given are atomic masses.

You can look up other nuclear data from websites
`https://www-nds.iaea.org/relnsd/vcharthtml/VChartHTML.html`
`http://www.nndc.bnl.gov/nudat2/`
`http://atom.kaeri.re.kr/nuchart/`
`http://people.physics.anu.edu.au/~ecs103/chart/`