Week 9 Assessed Questions Physical Cosmology

$$V(\phi) = e^{\lambda\phi}.$$

(a) Slow-roll approx. : $\frac{\dot{\phi}^2}{2} << |V(\phi)|$.

KG Eqn.: $\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0$

Slow roll: $\ddot{\phi} << |V|$, but $3H\dot{\phi} \sim |V|$

$\rightarrow$ KG Eqn. becomes: $3H\dot{\phi} \approx -\frac{dV}{d\phi}$

Friedman Eqn: $H^2 = \frac{8\pi G}{3}\rho_\phi$

and $\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi) \approx V(\phi)$

Slow roll: $H^2 \approx \frac{8\pi G}{3} V$.

Sub. in $V = e^{\lambda\phi}$:

$$3H\dot{\phi} \approx -\lambda e^{\lambda\phi} \; ; \; H^2 \approx \frac{8\pi G}{3} e^{\lambda\phi}.$$

(b) Solve to find $\phi(a)$.

First, write: $\dot{\phi} = \frac{d\phi}{dt} = \frac{d\phi}{da}\cdot\frac{da}{dt} = aH\cdot\frac{d\phi}{da}$

since $\frac{da}{dt} = aH$.

KG Eqn. becomes:

$$3aH^2 \frac{d\phi}{da} = -\lambda e^{\lambda\phi}$$

Sub. in Friedman Eqn:

$$H^2 \simeq \frac{e^{\lambda\phi}}{3} \qquad (\text{in units where } 8\pi G = 1)$$

$$\rightarrow 3a \cdot \frac{e^{\lambda\phi}}{3} \frac{d\phi}{da} = -\lambda e^{\lambda\phi}$$

Perform cancellations and rearrange:

$$a\frac{d\phi}{da} = -\lambda \quad \rightarrow \quad d\phi = -\lambda \frac{da}{a}$$

Integrate: $$\int_{\phi_i}^{\phi} d\phi = -\lambda \int_{a_i}^{a} \frac{da}{a}$$

$$\rightarrow \left[\phi\right]_{\phi_i}^{\phi} = -\lambda \left[\log a\right]_{a_i}^{a}$$

so $$\phi - \phi_i = -\lambda(\log a - \log a_i) = -\lambda \log\left(\frac{a}{a_i}\right).$$

$$\rightarrow \phi(a) = \phi_i + \log\left(\left(\frac{a}{a_i}\right)^{-\lambda}\right)$$

(c) $\Delta\phi = 0.6$. $\lambda = -0.01$

Plug into solution for $\phi(a)$:

$$\Delta\phi = \phi_f - \phi_i = \log\left(\left(\frac{a_f}{a_i}\right)^{-\lambda}\right) = 0.6$$

Exponentiate:

$$\left(\frac{a_f}{a_i}\right)^{-\lambda} = e^{0.6} \longrightarrow \frac{a_f}{a_i} = \left(e^{0.6}\right)^{-\frac{1}{\lambda}}$$

Plug in $\lambda = -0.01$:

$$\frac{a_f}{a_i} = \left(e^{0.6}\right)^{\frac{-1}{-0.01}} = \left(e^{0.6}\right)^{100} = \underline{e^{60}}$$

60 e-folds.

(d) $T_f \approx 10^{22}\,K$. at end of inflation

$T_0 \approx 2.7\,K$ today.

We know that $T = T_0(1+z)$

so $$\frac{T_f}{T_0} = \frac{T_0(1+z_f)}{T_0(1+0)} = 1 + z_f$$

and $a_f = \frac{1}{1+z_f}$, scale factor at end of inflation.

$$\rightarrow a_f = \frac{T_0}{T_f} = \frac{2.7\,K}{10^{22}\,K}.$$

No. of e-folds is no. of power of 'e' of expansion

$$\rightarrow \quad \frac{a_0}{a_f} = \frac{1}{a_f} = \frac{10^{22}}{2.7} = 3.7 \times 10^{21}$$

To get no. of e-folds, take the natural log:

$$\ln\left(3.7 \times 10^{21}\right) = \underline{49.7 \text{ e-folds}}$$

(e) If we want the same number of e-folds during inflation as after inflation:

Need $\frac{a_f}{a_i} = e^{49.7}$.

From part (c): $$\Delta\phi = \log\left(\left(\frac{a_f}{a_i}\right)^{-\lambda}\right)$$

$$= \log\left(\left(e^{49.7}\right)^{+0.01}\right)$$

$$= \log\left(1.644\right) = \underline{0.497}$$

So for this particular choice of λ, the field displacement $\Delta\phi = \frac{(\text{no. e-folds})}{100}$.