1. (a) Expand $y = x(1+x)^{-2}$ about $x \approx 0$.

$$y(x)\Big|_{x=0} \approx \underbrace{y(0)}_{=0} + \underbrace{\frac{dy}{dx}\Big|_{x=0}}\, x + \ldots$$

$$= (1+x)^{-2} - 2(1+x)^{-3}\cdot x$$

$$= (1+x)^{-2}\left(1 - \frac{2x}{1+x}\right)$$

Evaluate $\frac{dy}{dx}\Big|_{x=0} = 1\cdot\left(1 - 0\right) = 1$

$\rightarrow$ $\underline{y(x) \approx x}$ to linear order.

(b) Damped harmonic oscillator, $\ddot{x} + 2b\omega\dot{x} + \omega^2 x = 0$

(i) Simple harmonic oscillator has no damping term:

$$\ddot{x} + \omega^2 x = 0$$

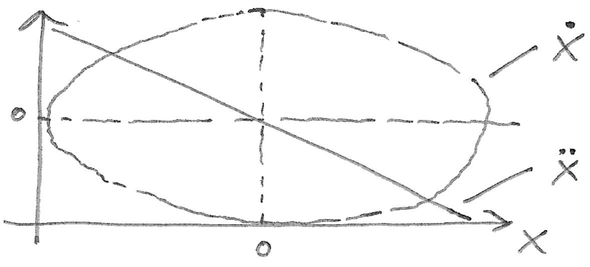
(ii) Solution is $x = A\cos(\omega t + \phi)$.

$$\rightarrow \dot{x} = -A\omega\sin(\omega t + \phi)$$

$$\ddot{x} = -\omega^2 x = -\omega^2 A\cos(\omega t + \phi)$$

As a function of position:

- ~~When $x = 0$,~~ $\dot{x}$ is $\pi/2$ out of phase and has amplitude $-\omega$ relative to x.
- $\ddot{x}$ is in phase with x but has amplitude $-\omega^2$ relative to x.



(iii) For $x \approx 0$, $\ddot{x}$ is also 0.
$\dot{x}$ is at its max./min value

→ Neglect $\ddot{x}$ in this case
$\dot{x}$ term at its largest
keep $\omega^2 x$ term, as ω is relatively large, so will become important as $|x|$ increases away from $x=0$.

For $x \approx \pm X_{max}$, $\dot{x}$ is zero
$\ddot{x}$ is at its max./min. with the opposite sign

→ $\dot{x}$ is small and $b \lesssim \omega$, so small in comparison to $\omega^2 x$ term; can probably neglect this term.
keep $\ddot{x}$ as it will be $\ddot{x} \approx -\omega^2 x$, so as large as the $\omega^2 x$ term.

(iv) If the damping was very strong, we would be far away from the simple harmonic oscillator case.
The damping term would almost always be important.
Could probably neglect the $\omega^2 x$ term; acceleration $\ddot{x}$ will be driven by the damping term $2b\omega\dot{x}$, solution will be an exponential decay of x.

2. $\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0.$

(a) Universe contains only scalar field.

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi)$$
$$P_\phi = \frac{\dot{\phi}^2}{2} - V(\phi)$$

Friedmann Eqn. : $H^2 = \frac{8\pi G}{3}\rho$ (flat, so $k=0$, and no cosmological constant.)

$$= \frac{8\pi G}{3}\rho_\phi$$
$$= \frac{8\pi G}{3}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right).$$

(b) Slow-roll approximation : $\frac{\dot{\phi}^2}{2} \ll |V(\phi)|$, i.e. kinetic energy of scalar field is small.

$$\Rightarrow \rho_\phi \approx V(\phi)$$
$$\rightarrow H^2 \approx \frac{8\pi G}{3} V$$ as required.

(c) Klein-Gordon Eqn. :

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0$$

This term could be large or small – can't neglect it.

$\dot{\phi}$ is small, so $\frac{d\dot{\phi}}{dt}$ must also be small (assuming $\dot{\phi}$ stays small for a while!)

$\dot{\phi}$ is small but H is large, so can't neglect this term.

KG Eqn. becomes: $3H\dot{\phi} \approx -\frac{dV}{d\phi}$ (slow-roll approx.)

(d) $\dot{\phi}$ was neglected in the Friedmann Eqn. (where it appeared as $\frac{\dot{\phi}^2}{2}$). It was not neglected in the KG equation (where it appeared as $3H\dot{\phi}$). This is because H is large in the early universe, so even if $\dot{\phi}$ is small, $H \times \dot{\phi}$ could be non-negligible.

3. Inflaton as a 'clock' field.

Relate $d\phi$ to dt under slow-roll assumption.

KG eqn: $3H\dot{\phi} \approx -\frac{dV}{d\phi}$

Now we need to be careful which $d\phi$ and which dt we use!

KG: $\frac{3}{a}\frac{da}{dt} \cdot \frac{d\phi}{dt} = -\frac{dV}{d\phi}$.

But $\frac{1}{a}\frac{da}{dt} \simeq \text{Const.} \simeq \sqrt{\frac{8\pi G V}{3}}$

and $\frac{dV}{d\phi}$ is something we can calculate easily for any model.

$\rightarrow 3H\frac{d\phi}{dt} = \cancel{\sqrt{24\pi}} \sqrt{24\pi G}\sqrt{V}\frac{d\phi}{dt} = -\frac{dV}{d\phi}$

$\Rightarrow dt = -\frac{\sqrt{V}}{dV/d\phi} \times \sqrt{24\pi G}\, d\phi$

4. Hubble radius during inflation.

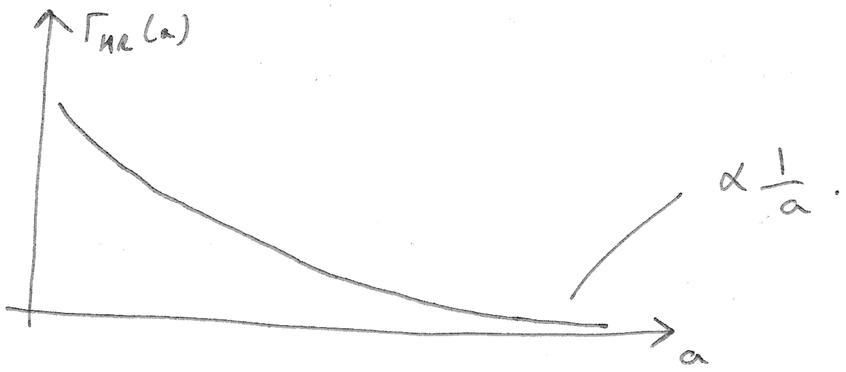
During slow-roll inflation, $\frac{\dot\phi^2}{2} << |V|$ (using units where $c=1$)

$$w_\phi = \frac{P_\phi}{\rho_\phi} = \frac{\frac{\dot\phi^2}{2} - V}{\frac{\dot\phi^2}{2} + V} \simeq \frac{-V}{V} \simeq -1$$

We know that $a(t) \propto e^{Ht}$ when $w=-1$ (exponential expanding solution).

$r_{HR}(a) \propto (aH)^{-1}$; $a \propto e^{Ht}$ and $H \simeq$ const.

$$\Rightarrow r_{HR} \propto \frac{1}{a \times \text{const.}} \propto a^{-1}$$



So the Hubble radius shrinks.

Inflation solves the <u>horizon problem</u> by exponentially expanding a smooth, homogeneous patch of the early universe into a large patch that fills the Hubble radius/horizon (which gets smaller). So, within our patch of the universe, the mass/energy distribution is very close to homogeneous by the end of inflation, and regions that are inhomogeneous are far outside our Hubble radius.

It solves the <u>monopole problem</u> in a similar way. If there were many monopoles/topological ~~defects~~ defects before inflation, the massive amount of expansion during inflation will have completely diluted their number density, so only a handful would be present within our Hubble radius at most.

5. Flatness problem.

(a) $\Omega_0(t) = \dfrac{\rho_{tot}(t)}{\rho_{cr}(t)}$. $\rho_{cr}(t) = \dfrac{3H^2}{8\pi G}$ (Friedmann Eqn. with $k=0$).

and $H^2 = \dfrac{8\pi G}{3}\rho_{tot}(t) - \dfrac{kc^2}{a^2}$.

Divide Friedmann Eqn. by H^2:

$$\underbrace{\frac{H^2}{H^2}}_{=1} = \underbrace{\frac{8\pi G}{3H^2}}_{=\frac{1}{\rho_{cr}(t)}} \cdot \rho_{tot}(t) - \frac{kc^2}{a^2H^2}$$

$$\rightarrow 1 = \frac{\rho_{tot}(t)}{\rho_{cr}(t)} - \frac{kc^2}{(aH)^2} = \Omega_{tot}(t) - \frac{kc^2}{(aH)^2}$$

$$\Rightarrow |1 - \Omega_{tot}(t)| = \frac{|k|c^2}{(aH)^2} \propto (aH)^{-2}$$ as required.

(b) $\Omega_k = +10^{-3}$ (negatively curved)

$\Omega_r \approx 1$.

Since curvature is almost negligible, we can use:

$$H^2 = H_0^2 \overset{1}{\cancel{\Omega_r}} a^{-4} \simeq H_0^2 a^{-4} \Rightarrow H = H_0 a^{-2}.$$

From above, $1 - \Omega_{tot}(t) = -\dfrac{kc^2}{(aH)^2}$. Recall that $\underline{-kc^2 = H_0^2 \Omega_k}$

$$\Rightarrow 1 - \Omega_{tot}(t) = \frac{H_0^2 \Omega_k}{a^2 H^2} \approx \frac{\cancel{H_0^2} \Omega_k}{a^2 \cdot \cancel{H_0^2} a^{-4}} = \underline{\Omega_k a^2}$$

Using $1 - \Omega_{tot}(t) \simeq \Omega_k a^2$ for this universe:

(i) $a = 0.1 \Rightarrow 1 - \Omega_{tot} = (0.1)^2 \times 10^{-3} = \underline{10^{-5}}$

(ii) $a = 10^{-3} \Rightarrow 1 - \Omega_{tot} = 10^{-6} \times 10^{-3} = \underline{10^{-9}}$

(iii) $a = 10^{-10} \Rightarrow 1 - \Omega_{tot} = 10^{-20} \times 10^{-3} = \underline{10^{-23}}$

(c) Inflation ended at $T \approx 10^{22}$ K.

$T_0 = 2.725$ K.

We know that temperature scales with redshift as:

$$T(z) = T_0 (1+z)$$

$$\rightarrow 1 + z_{end} = \frac{T(z)}{T_0} = \frac{10^{22}}{2.725} = 3.67 \times 10^{21}$$

$$a_{end} = \frac{1}{1+z_{end}} = \underline{2.725 \times 10^{-22}}$$ scale factor at end of inflation.

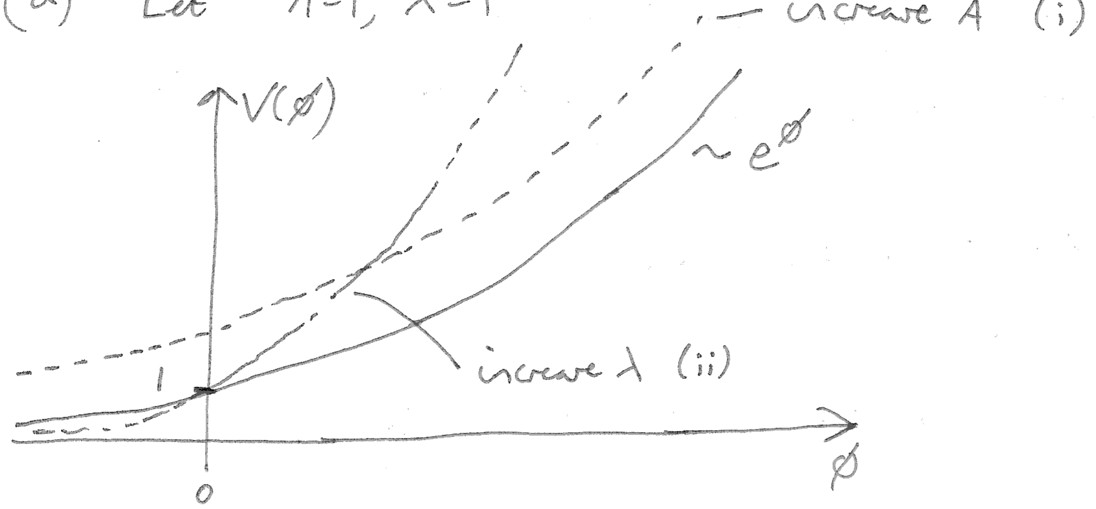
(d) If the universe is reasonably close to flat today (e.g. $\Omega_k \approx 10^{-3}$) this means that it must be <u>extremely</u> close to flat in the early universe. It seems highly improbable that a tiny deviation from flatness (e.g. 10^{-23} at $a = 10^{-10}$) would have occurred by chance. This is the <u>flatness problem</u> – for the universe to be close to flat today it must be fine-tuned to be <u>extremely</u> close to flat in the past.

Inflation solves this problem by driving $1 - \Omega_{tot}(t)$ to very low values, since $(aH)^{-2} \propto a^{-2}$ during accelerating expansion.

6. Exponential Potential

$V = Ae^{\lambda\phi}$

(a) Let $A=1$, $\lambda=1$

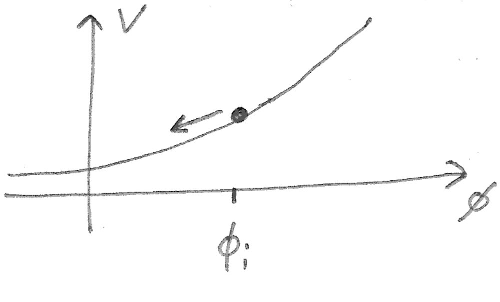


(b) Kinetic energy: $\frac{\dot{\phi}^2}{2}$ (does not involve the potential)

Pressure: $P_\phi = \frac{\dot{\phi}^2}{2} - V(\phi) = \frac{\dot{\phi}^2}{2} - Ae^{\lambda\phi}$

Density: $\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi) = \frac{\dot{\phi}^2}{2} + Ae^{\lambda\phi}$

(c) Let $\phi_i = 1$, $\dot{\phi}_i = 0$.



The scalar field starts at rest (i.e. with $\dot{\phi}=0$)

$\Rightarrow$ Pressure $= -V(\phi_i)$

Density $= +V(\phi_i)$

K.E $= 0$

Eqn. of State $w_\phi = \frac{P_\phi}{\rho_\phi} = -1$

As field starts to move, it will go in the direction of <u>decreasing</u> ϕ, "falling" down the potential. $\Rightarrow \dot{\phi} < 0$. But, all quantities above depend on $\dot{\phi}^2$.

- KE grows increasingly +ve.
- Density may increase or decrease ($\frac{\dot{\phi}^2}{2}$ increases, but $V(\phi)$ decreases as $\phi \to 0$). If we worked out the details, we would find that the density decreases (see below).
- Pressure may also increase or decrease.

Let's analyse the kinetic energy as a small value compared to the potential: $\epsilon(\phi) = \dfrac{\dot{\phi}^2}{2V(\phi)}$, a new "small parameter".

Eqn. of state becomes:

$$w_\phi = \frac{\frac{\dot{\phi}^2}{2} - V(\phi)}{\frac{\dot{\phi}^2}{2} + V(\phi)} = \frac{\epsilon - 1}{\epsilon + 1}.$$

If we assume that the potential is +ve (it doesn't have to be!), we find $\epsilon \geq 0$. This implies that when $\frac{\dot{\phi}^2}{2} > 0$, $\epsilon > 0$, and so $w_\phi > -1$.

$w = -1$ is the limiting case of a cosmological constant (constant density). Anything with $w > -1$ must therefore have a <u>decreasing</u> density with time, due to cosmic expansion.