Week 9: Inflation

Please hand in the completed problems by **Thursday 5th of December at 4pm**. *Please show your working and write neatly, staple all sheets together, and write your name and student number at the top of the first sheet.*

1. **Maths practice: Neglecting small terms**
   (a) By performing a Taylor expansion, expand $y = x\,(1+x)^{-2}$ to linear order in the small parameter $x \simeq 0$.
   (b) A damped harmonic oscillator has an equation of motion $\ddot{x} + 2b\omega\dot{x} + \omega^2 x = 0$. We will assume that the damping is positive, $b > 0$, but relatively weak, $b \lesssim \omega$.
      i. Using the same notation as above, write down the equation of motion for a *simple* harmonic oscillator.
      ii. Sketch graphs of $\dot{x}$ and $\ddot{x}$ as a function of position, x, for a simple harmonic oscillator.
      iii. By considering how $\ddot{x}$, $\dot{x}$, and x behave as a function of x for a simple harmonic oscillator, suggest which of the terms in the equation of motion for the damped harmonic oscillator can be neglected when (i) $x \approx 0$; (ii) $x \approx \pm x_{\rm max}$.
      iv. How would your answers change if the damping was very strong, $b \gg \omega$?

2. **Slow-roll approximation**
   The cosmological Klein-Gordon equation for a scalar field is

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0.$$

   (a) Use the Friedmann equation to find an expression for H^2 in a flat universe containing only a scalar field ϕ.
   (b) Show that $H^2 \approx (8\pi G/3)\,V(\phi)$ in the slow-roll approximation.
   (c) Write down the Klein-Gordon equation in the slow-roll approximation.
   (d) You should notice that one quantity that was neglected in the Friedmann equation still remains in the Klein-Gordon equation. Explain why this is.

3. **The inflaton as a 'clock' field**
   If the inflaton field only travels in one direction (i.e. $\dot{\phi}$ is always positive or always negative), we can use it as a time coordinate. This is analogous to the way that we use the scale factor a as a time coordinate because it increases monotonically in an expanding universe.

   Using the slow-roll approximation of the cosmological Klein-Gordon equation, find an expression relating the infinitessimal time interval dt to the infinitessimal field displacement $d\phi$.

4. **Hubble radius during inflation**
   Sketch how the Hubble radius $r_{\rm HR}(a)$ depends on the scale factor during slow-roll inflation. Use your sketch to explain how inflation solves the horizon and monopole problems.

5. **Flatness problem**
   If the Universe is exactly flat, the total fractional energy density parameter $\Omega_0 = 1$.
   (a) Show that $|1 - \Omega_0| \propto (aH)^{-2}$.
   (b) Consider a Universe that is very slightly negatively curved today, $\Omega_k = 10^{-3}$, and that is otherwise radiation-dominated ($\Omega_r \approx 1$). Evaluate $1 - \Omega_0$ at:
   (i) $a = 0.1$; (ii) $a = 10^{-3}$; (iii) $a = 10^{-10}$ for this Universe.
   (c) Inflation ended when the Universe was at a temperature of $T \approx 10^{22}$ K. Assuming that $T_0 = 2.725$ K today, calculate the approximate scale factor corresponding to the end of inflation.
   (d) Use your answers above to briefly explain what the flatness problem is, and how inflation solves it.

(PLEASE TURN OVER)

6. **Exponential potential**
   Consider a scalar field ϕ that has a potential of the form $V(\phi) = Ae^{\lambda\phi}$.

   (a) Sketch the potential $V(\phi)$ for $A = 1$ and $\lambda = 1$, including positive and negative values of ϕ.
   On the same sketch, show how $V(\phi)$ would change if you increased the values of (i) A, and (ii) λ.

   (b) Write expressions for the kinetic energy, pressure, and density of the scalar field in terms of $\dot{\phi}$ and this potential (or its derivatives).

   (c) Consider initial conditions for the scalar field $\phi_i = 1$, $\dot{\phi}_i = 0$. Describe the qualitative behaviour of the kinetic energy, pressure, density, and equation of state of the scalar field immediately after it starts to move. Which direction will the field begin to move in?

Assessed question: Scale factor during inflation

Consider a scalar field with potential $V(\phi) = e^{\lambda\phi}$ that is the only matter/energy field in a flat, expanding universe.

(a) Write down the Klein-Gordon and Friedmann equations for this cosmological model under the slow-roll approximation. Make sure to substitute in the definition of $V(\phi)$ from above.
[5 marks]

(b) Solve these equations to find an expression for the scalar field as a function of the scale factor, $\phi(a)$. You may use units where $8\pi G = 1$, and the initial condition $\phi(a_i) = \phi_i$.
[7 marks]

(c) Consider a situation in which the inflaton moves a distance $\Delta\phi = 0.6$ between the beginning and end of inflation. Calculate the ratio of the final to initial scale factors, a_f/a_i, for this model if $\lambda = -0.01$.
[5 marks]

(d) Inflation is thought to have ended when the Universe reached a temperature of $T \approx 10^{22}$ K. Calculate how many e-folds of expansion have occurred between the end of inflation and today (when $T \approx 2.7$ K).
[4 marks]

(e) Calculate the value of $\Delta\phi$ that would be required for as many e-folds of expansion to have happened *during* inflation as have happened *since* inflation in this inflationary model, assuming that $\lambda = -0.01$.
[4 marks]