Queen Mary
University of London

BSc/MSci Examination by Course Unit

Late Summer Examination Period 2019

SPA6311 Physical Cosmology Duration: 2 hours 30 minutes

YOU ARE NOT PERMITTED TO READ THE CONTENTS OF THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR.

Instructions:

Answer ALL questions from Section A. Answer ONLY TWO questions from Section B. Section A carries 50 marks, each question in section B carries 25 marks.

If you answer more questions than specified, only the first answers (up to the specified number) will be marked. Cross out any answers that you do not wish to be marked.

Only non-programmable calculators are permitted in this examination. Please state on your answer book the name and type of machine used.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

Important note: The academic regulations state that possession of unauthorised material at any time when a student is under examination conditions is an assessment offence and can lead to expulsion from QMUL.

Please check now to ensure you do not have any notes, mobile phones or unauthorised electronic devices on your person. If you have any, raise your hand and give them to an invigilator immediately. It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption is also an assessment offence.

EXAM PAPERS MUST NOT BE REMOVED FROM THE EXAM ROOM.

Examiners:
Dr. A. Pourtsidou
Dr. K. A. Malik

SECTION A **Answer ALL questions in Section A**

Questions that are *not* applicable to the 2019/20 syllabus are highlighted in red.

Question A1

a) Define the term *"Nucleosynthesis"* as used in cosmology.

b) Give an estimate for the times Nucleosynthesis started and ended.

c) Show that during Nucleosynthesis the temperature of the universe, T, depends on time t as $T \propto t^{-1/2}$.

[10 marks]

Question A2

a) Use the Hubble parameter today, H_0, to calculate a rough estimate of the age of the universe. You can assume that $h = 0.7$. Explain why this is only a rough estimate.

b) Using the definition of the Hubble parameter in general, $H = \dot{a}/a$, give the exact expression for the age of the universe as an integral involving only the scale factor and the Hubble parameter.

[8 marks]

Question A3

a) Sketch in the same diagram a galaxy rotation curve as observed in nature, and a rotation curve as predicted from Newtonian dynamics.

b) Briefly describe how the difference between the predicted and the observed rotation curves can be explained.

c) Give two examples for dark matter candidates.

[10 marks]

Question A4

Name two of the main epochs in the history of the universe, and state how the energy density evolves in these epochs in terms of the scale factor.

[4 marks]

Question A5

a) Estimate the mass density of galaxies in the universe in SI units, given a typical separation of 1Mpc and a typical mass of $10^{11}M_\star$, [you may assume $1M_\star \simeq 2\times 10^{30}$kg].

b) Calculate the energy density of the radiation left over from the CMB (i) today and (ii) at redshift $z=4$.

[8 marks]

Question A6

Derive the acceleration [Raychaudhuri] equation, by using the Friedmann and the energy conservation equations, or by other means.

[6 marks]

Question A7

In a closed Friedmann-Robertson-Walker universe, luminosity distance, d_{lum}, and proper distance, d_{p}, are related by

$$d_{\mathsf{lum}} = \frac{1+z}{\sqrt{k}}\sin(\sqrt{k}\,d_{\mathsf{p}})\,,$$

where z is the redshift and k the curvature constant. Show that for small redshifts and in the nearby universe

$$d_{\mathsf{lum}} \approx d_{\mathsf{p}}\,.$$

[4 marks]

SECTION B **Answer TWO questions from Section B**

Question B1

Assume that the evolution equation for the density contrast δ is given by

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - \frac{\alpha}{a^2}\nabla^2\delta - \beta\bar{\rho}\delta = 0\,,$$

where α and β are positive constants, and $\bar{\rho}$ is the energy density in the background.

a) In Fourier space we can replace Laplacians by $\nabla^2 \to -k^2$, where k is the comoving wave number. Make this replacement and hence read off the physical Jeans wave number from the evolution equation above, and give the Jeans length, λ_J.

Briefly discuss the meaning of the Jeans length.

[9 marks]

b) In the case of a flat universe dominated by radiation, solve the evolution equation for the density contrast given above in the large scale limit, that is for $k \to 0$. Write down the time evolution of the growing and the decaying modes.

[9 marks]

c) It can be shown that the evolution of the density contrast of the cold dark matter, δ_{cdm}, during radiation domination is governed by

$$H^{-2}\ddot{\delta}_{\text{cdm}} + H^{-1}\dot{\delta}_{\text{cdm}} = 0\,.$$

Solve the evolution equation of δ_{cdm} for a flat, radiation dominated FRW universe.

[7 marks]

Question B2

a) State expressions for the energy density of the curvature term, ρ_k, and the energy density of the cosmological constant term, ρ_Λ, in the Friedmann equation, so that the right hand side of the Friedmann equation can be written as the sum of energy densities.

[2 marks]

b) Using the Friedmann equation in the appendix as your starting point, derive the Friedmann equation in the general case in terms of density parameters evaluated at the present day, for a universe containing radiation, matter, a curvature term, and a cosmological constant. Use the scale factor a as the time variable, assuming the value of the scale factor today is $a(t_0) = 1$. Give expressions for the present day density parameter for the cosmological constant, $\Omega_{\Lambda,0}$, and the present day density parameter for the curvature term, $\Omega_{\mathsf{k},0}$, in terms of Λ, k, and H_0.

[10 marks]

c) Show that the Friedmann equation derived in question b) can be written as

$$1 = \Omega_\mathsf{m}(a) + \Omega_\mathsf{r}(a) + \Omega_\mathsf{k}(a) + \Omega_\Lambda(a)\,.$$

[3 marks]

d) Now assume an open universe with no cosmological constant. You can neglect the matter and radiation content present. At late times, give an expression for the scale factor a in terms of time t. State why the matter content of the universe is not important in this case.

[10 marks]

Question B3

a) The Friedmann equation can be written in terms of density parameters as

$$1 - \Omega_{\mathsf{tot}} - \Omega_\Lambda = \frac{1}{(aH)^2}\,.$$

Define all symbols in the equation.

[4 marks]

b) Briefly explain the *flatness problem* in cosmology.

[4 marks]

c) How does the right hand side of the equation above evolve during

1. radiation domination;
2. matter domination.

Briefly state the implication this has for the flatness problem.

[7 marks]

d) Derive the time evolution of the scale factor in a de Sitter universe (a flat, cosmological constant dominated universe). Use your results to show how inflation solves the flatness problem.

[7 marks]

e) Briefly define the term *inflation* as used in cosmology, and state the condition on the scale factor.

[3 marks]

Question B4

a) In this question we use units where $c = 1$.

Consider a matter-dominated open universe. Show, by substitution, that

$$a(\psi) = \frac{4\pi G\rho_0}{3|k|}(\cosh\psi - 1)\,,$$
$$t(\psi) = \frac{4\pi G\rho_0}{3|k|^{\frac{3}{2}}}(\sinh\psi - \psi)\,.$$

is a solution to the Friedmann equation, where ρ_0 is the value of the matter density when $a = 1$, k is the curvature constant, and ψ a parameter.

[12 marks]

Hint: recall that $\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}y}/\frac{\mathrm{d}x}{\mathrm{d}y}$.

b) Show that in a FRW universe the proper distance to a galaxy at coordinate (r_{g}, 0, 0) from an on observer at (0,0,0) is given by

$$d_{\mathsf{p}}(t) = a(t)\int_0^{r_{\mathrm{g}}} \frac{dr}{\sqrt{1-kr^2}}\,.$$

Find $d_{\mathsf{p}}(t)$ of this galaxy in the flat case.

[6 marks]

c) Derive Hubble's law using your results in the previous question.

[7 marks]

End of Paper - An Appendix of 1 page follows

Appendix

Useful Data and Equations

Except where specifically stated, the following results may be quoted without proof:

The Friedmann-Robertson-Walker metric is

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] ,$$

where $a(t)$ is the scale factor, k is the curvature constant and r, θ, ϕ are spherical coordinates. The equations governing the evolution of the scale factor in a homogeneous and isotropic universe are:

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} , \qquad \text{(Friedmann equation)}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3} , \qquad \text{(Acceleration equation)}$$

where ρ is the mean density of the universe, P is the pressure, and Λ is the cosmological constant. The energy conservation equation is given by

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0 .$$

The density parameter Ω and the critical density ρ_{c} are defined as

$$\Omega = \frac{\rho}{\rho_{\mathsf{c}}} , \qquad \rho_{\mathsf{c}} = \frac{3H^2}{8\pi G} .$$

The deceleration parameter q is defined as

$$q = -\frac{a\ddot{a}}{\dot{a}^2} .$$

By convention, the subscript 0 on any of these parameters denotes the value at the present time.

Additional constants

	1Mpc	$= 3.09 \times 10^{22}$ m
Critical density today	$\rho_{\mathsf{c},0}$	$= 1.88h^2 \times 10^{-26}$ kg m^{-3}
Hubble parameter today	H_0	$= 100h$ km s^{-1} Mpc^{-1}
Temperature of the CMB today	T_0	$= 2.73$ K
Radiation density constant	σ_{rad}	$= 7.6 \times 10^{-16}$ J m^{-3} K^{-4}
Speed of light in vacuum	c	$= 3.00 \times 10^8$ m s^{-1}