Queen Mary
University of London

BSc/MSci Examination by Course Unit

Main Examination Period 2019

SPA6311 Physical Cosmology Duration: 2 hours 30 minutes

YOU ARE NOT PERMITTED TO READ THE CONTENTS OF THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR.

Instructions:

Answer ALL questions from Section A. Answer ONLY TWO questions from Section B. Section A carries 50 marks, each question in section B carries 25 marks.

If you answer more questions than specified, only the first answers (up to the specified number) will be marked. Cross out any answers that you do not wish to be marked.

Only non-programmable calculators are permitted in this examination. Please state on your answer book the name and type of machine used.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

Important note: The academic regulations state that possession of unauthorised material at any time when a student is under examination conditions is an assessment offence and can lead to expulsion from QMUL.

Please check now to ensure you do not have any notes, mobile phones or unauthorised electronic devices on your person. If you have any, raise your hand and give them to an invigilator immediately. It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption is also an assessment offence.

EXAM PAPERS MUST NOT BE REMOVED FROM THE EXAM ROOM.

Examiners:
Dr. A. Pourtsidou
Dr. K. A. Malik

SECTION A Answer ALL questions in Section A

Questions that are *not* applicable to the 2019/20 syllabus are highlighted in red.

Question A1

a) Define the term *recombination* as used in cosmology.

b) Define the term *Epoch of Reionization* as used in cosmology. State whether the Epoch of Reionization occurred before or after recombination.

c) Give the mathematical expression for the redshift, z, in terms of the wavelength, λ.

[8 marks]

Question A2

a) Briefly explain the terms *absolute luminosity* and *apparent luminosity* as used in astronomy and cosmology.

b) A star has absolute luminosity L. State the mathematical formula that gives its apparent luminosity, ℓ, at distance R, assuming Euclidean geometry.

[6 marks]

Question A3

a) Define the length unit *parsec*.

b) State whether we expect the universe to be homogeneous and isotropic on Gpc scales.

c) Briefly explain what is meant by the *cosmic distance ladder*.

[6 marks]

Question A4

a) Show that for an open universe with $k < 0$ the proper distance, which is defined in general as

$$d_{\mathsf{p}} = \int_0^{r_0} \frac{\mathrm{d}r}{\sqrt{1-kr^2}} \, ,$$

is given by

$$d_{\mathsf{p}} = \frac{1}{\sqrt{|k|}} \sinh^{-1}(\sqrt{|k|} r_0) \, .$$

Hint: You will need the identity $\cosh^2 y - \sinh^2 y = 1$.

[10 marks]

Question A5

a) State one piece of observational evidence for dark matter, and one for dark energy.

b) State one difference between the physical properties of dark matter and dark energy.

[5 marks]

Question A6

a) Name the three different geometries the universe can have (assuming that space is homogeneous and isotropic), and state the values of the curvature parameter k for each one. State which geometry is supported by observational data.

b) For each geometry sketch a 2-dimensional analogous surface. State the sum of the interior angles of a triangle for each case.

[8 marks]

Question A7

a) Give an example of a relativistic particle and an example of a non-relativistic particle.

b) Briefly explain why neutrinos cannot account for the *cold* dark matter component of the universe.

c) Briefly describe the meaning of *dark matter halo* as used in cosmology.

[7 marks]

SECTION B **Answer TWO questions from Section B**

Question B1

a) Define the equation of state parameter w as used in cosmology, and state the value of w for pressureless matter, and the value of w for radiation.

[4 marks]

b) Using the energy conservation equation in the Appendix in $c = 1$ units, show that $\rho = \text{const}$ for a fluid with $w = -1$ (that is, the cosmological constant).

[2 marks]

c) Assuming a flat universe containing pressureless matter and radiation, explain why we can neglect the effects of radiation at late times.

[4 marks]

d) Show that in a flat universe dominated by a fluid with equation of state parameter w, the deceleration parameter q (see Appendix) is given by

$$q = \frac{1}{2}(1 + 3w).$$

Derive the range of w values for which we get accelerated expansion.

[7 marks]

e) Assume a radiation dominated flat universe. Solve the Friedmann equation to find the Hubble rate a as a function of time t (with $a = 0$ at the time of the Big Bang, and $a = 1$ today). Calculate what the age of the Universe would be according to this model.

[8 marks]

Question B2

The evolution equation for the density contrast δ is given by

$$\ddot{\delta} + 2H\dot{\delta} - \frac{c_\mathsf{s}^2}{a^2}\nabla^2\delta - 4\pi G\bar{\rho}\delta = 0\,,$$

where $H = \frac{\dot{a}}{a}$ is the Hubble rate, c_s^2 is the adiabatic sound speed and $\bar{\rho}$ is the energy density in the background.

a) Explain what is meant by *density contrast* in cosmology, and state the mathematical formula that defines it.

[3 marks]

b) In Fourier space we can replace Laplacians by $\nabla^2 \to -k^2$, where k is the comoving wave number. Make this replacement and hence read off the Jeans wave number from the evolution equation above, and give the physical Jeans length, λ_J.

[6 marks]

Briefly discuss the meaning of the Jeans length.

[2 marks]

c) Assume that a flat universe is dominated by a cosmological fluid that results in a Hubble rate $H = t^{-1}$. Write down and solve the evolution equation for the density contrast δ in the large scale limit (that is for $k \to 0$) in this universe.

[6 marks]

d) Write down and solve the evolution equation for the *matter density contrast* δ_m in the large scale limit (that is for $k \to 0$) during the Λ-dominated era, and discuss the physical meaning of the results. You can assume $H = H_\Lambda = \mathsf{const}$, and that $H_\Lambda^2 \gg 4\pi G\bar{\rho}_m$.

[8 marks]

Question B3

a) Show that the rotation speed, $v(r)$, of a particle which is moving in a circular orbit of radius r within a spherical mass distribution is given by:

$$v(r) = \sqrt{\frac{GM(< r)}{r}} \,,$$

where $M(< r)$ is the mass within radius r.

[5 marks]

b) The dark matter halos which form in numerical simulations have density profiles, $\rho(r)$, which are well approximated by a Navarro-Frenk-White (NFW) profile

$$\rho(r) = \frac{\rho_0}{\left(\frac{r}{r_\mathsf{s}}\right)\left(1 + \frac{r}{r_\mathsf{s}}\right)^2} \,,$$

where r_s, is the scale radius, the radius at which $\mathrm{d}\ln\rho/\mathrm{d}\ln r = -2$.
Show that in this case the rotation curve, $v(r)$, has the form:

$$v(r) = \left\{4\pi G\rho_0 r_\mathsf{s}^2 \left(\frac{r_\mathsf{s}}{r}\right)\left[\ln\left(\frac{r_\mathsf{s}+r}{r_\mathsf{s}}\right) - \frac{r}{r_\mathsf{s}+r}\right]\right\}^{1/2} .$$

[12 marks]

c) Derive the behaviour of $\rho(r)$ and $v(r)$ in the limit $r \ll r_\mathsf{s}$.

[8 marks]

Question B4

a) State the *monopole problem* of the Big Bang model and describe how inflation solves it.

[5 marks]

b) Write down the form of the scale factor a as a function of time t during inflation.

[2 marks]

c) Define the number of e-foldings, N, in terms of the Hubble rate during inflation, the time inflation started, t_i, and the time inflation ended, t_f.

[4 marks]

d) Using $c = 1$ units, the *inflaton* field ϕ has energy density

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,,$$

and pressure

$$P = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,,$$

where $V(\phi)$ is the potential. Starting from the acceleration equation and setting $\Lambda = 0$, show that if the inflaton field is *slowly moving* we get inflation.

[6 marks]

e) Starting from the energy conservation equation for the inflaton field and using $c = 1$ units and the expressions for ρ and P in part B4)d), show that the equation for the field's evolution is

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi}\,.$$

[8 marks]

End of Paper - An Appendix of 1 page follows

Appendix

Useful Data and Equations

Except where specifically stated, the following results may be quoted without proof:
The Friedmann-Robertson-Walker metric is

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right],$$

where $a(t)$ is the scale factor, k is the curvature constant and r, θ, ϕ are spherical coordinates.
The equations governing the evolution of the scale factor in a homogeneous and isotropic universe are:

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}, \qquad \text{(Friedmann equation)}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3}, \qquad \text{(Acceleration equation)}$$

where ρ is the mean density of the universe, P is the pressure, and Λ is the cosmological constant. The energy conservation equation is given by

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0.$$

The density parameter Ω and the critical density ρ_{c} are defined as

$$\Omega = \frac{\rho}{\rho_{\mathsf{c}}}, \qquad \rho_{\mathsf{c}} = \frac{3H^2}{8\pi G}.$$

The deceleration parameter q is defined as

$$q = -\frac{a\ddot{a}}{\dot{a}^2}.$$

The subscript "0" on any of these parameters denotes the value at the present time.

Additional constants

	1Mpc	$= 3.09 \times 10^{22}$m
Critical density today	$\rho_{\mathsf{c},0}$	$=1.88h^2 \times 10^{-26}$ kg m^{-3}
Hubble parameter today	H_0	$=100h$ km s^{-1} Mpc^{-1}
Temperature of the CMB today	T_0	$= 2.73$ K
Radiation density constant	σ_{rad}	$= 7.6 \times 10^{-16}$ J m^{-3} K^{-4}
Speed of light in vacuum	c	$= 3.00 \times 10^8$ m s^{-1}