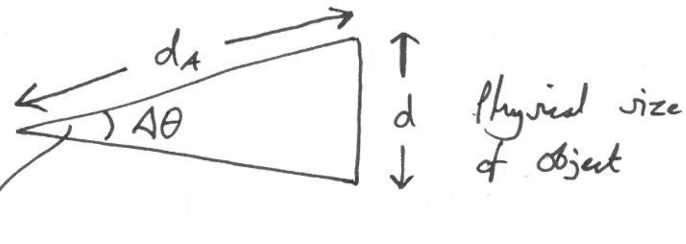
Week 4 — Assessed Question — Physical Cosmology

Flat, matter-only universe

$\Rightarrow \Omega_k = 0, \ \Omega_m = 1.$

(a) Definition of angular diameter distance:

$$\Delta\theta \simeq \frac{d}{d_A(z)}$$



(defined using the small-angle approx.)

Angular size of object

[2 MARKS] for correct definition.

Comoving distance to the object $= r(z)$.

$$d_A(z) = \frac{r(z)}{1+z}. \qquad [2 \text{ MARKS}]$$

(b) Calculate $d_A(z)$ in flat, matter-only universe.

Use $d_A = \dfrac{r}{1+z}$. ~~[1 MARK]~~

First, calculate $r(z)$:

Use FLRW line element for photons ($ds=0$)

$$\Rightarrow \quad 0 = -c^2 dt^2 + a^2 dr^2$$

so $c\,dt = \pm a\,dr$ [2 MARKS]

Use Friedmann Eqn. for matter-only universe:

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 = H_0^2 \cancelto{1}{\Omega_m} a^{-3}$$

Square root and rearrange:

$$\frac{da}{a} \cdot a^{3/2} = \pm H_0\, dt \qquad [2 \text{ MARKS}]$$

Sub. in to line element:

$$\frac{c}{H_0} a^{\frac{1}{2}} da = \pm a dr$$

$$\rightarrow \frac{c}{H_0} a^{-\frac{1}{2}} da = \pm dr$$

Integrate from the observer (us, today, at $a=1$, $r=0$):

$$\frac{c}{H_0} \int_{a=1}^{a} a^{-\frac{1}{2}} da = \pm \int_{r=0}^{r} dr$$

Solution:

$$\frac{c}{H_0} \left[2 a^{\frac{1}{2}} \right]_1^a = \pm \left[r \right]_0^r$$

$$\rightarrow r(a) = \pm \frac{2c}{H_0} \left(a^{\frac{1}{2}} - 1 \right)$$

choose $-$ve branch, as we want $r(z)$ to increase with z.

Sub. in $a = (1+z)^{-1}$:

$$r(z) = \frac{2c}{H_0} \left(1 - (1+z)^{-\frac{1}{2}} \right)$$

[2 MARKS]

Now use $d_A = \frac{r(z)}{1+z}$:

$$d_A(z) = \frac{2c}{H_0} \left((1+z)^{-1} - (1+z)^{-\frac{3}{2}} \right)$$

$$= \frac{2c}{H_0} \frac{1}{1+z} \left(1 - \frac{1}{\sqrt{1+z}} \right)$$

[1 MARK]

(c) Find Maxima of $d_A(z)$.

$$d_A(z) = \frac{2c}{H_0} \frac{1}{1+z} \left(1 - (1+z)^{-\frac{1}{2}} \right)$$

Maxima / minima occur when $\frac{d(d_A)}{dz} = 0$ [1 MARK]

Differentiate:

$$\frac{d(d_A)}{dz} = \frac{2c}{H_0} \left[\frac{-1}{(1+z)^2} \left(1 - (1+z)^{-\frac{1}{2}} \right) + \frac{1}{1+z} \left(+ \frac{1}{2} (1+z)^{-\frac{3}{2}} \right) \right]$$ [3 MARKS]

set derivative $= 0$: ($2c/H_0$ divides out)

$$\frac{1}{1+z} \cdot \frac{1}{2} \cdot (1+z)^{-\frac{3}{2}} = \frac{1}{(1+z)^2} \left(1 - (1+z)^{-\frac{1}{2}} \right)$$

Multiply both sides by $(1+z)^2$:

$$\frac{1}{2} (1+z)^{-\frac{1}{2}} = \left(1 - (1+z)^{-\frac{1}{2}} \right)$$

Multiply by $(1+z)^{\frac{1}{2}}$ and rearrange:

$$\frac{1}{2} = (1+z)^{\frac{1}{2}} - 1 \rightarrow (1+z)^{\frac{1}{2}} = 3/2$$

Square and rearrange:

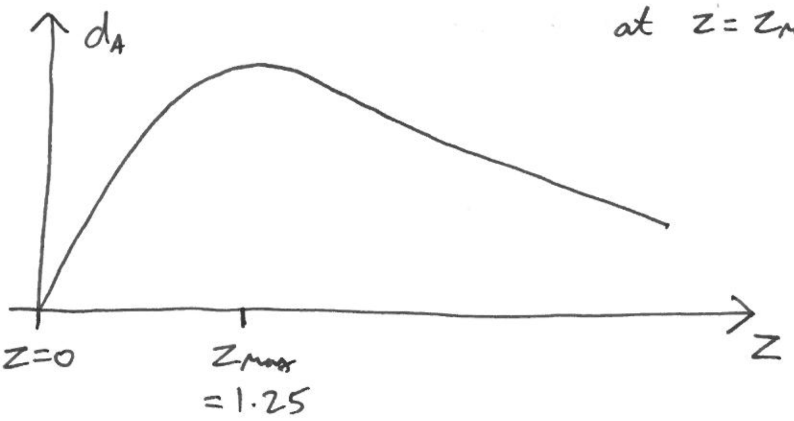
$$1 + z = 9/4$$

$$\Rightarrow z_{max} = 9/4 - 1 = 5/4 = 1.25$$ [3 MARKS]

(d) Object of fixed physical size, d.

$$\Delta\theta = \frac{d}{d_A(z)}$$

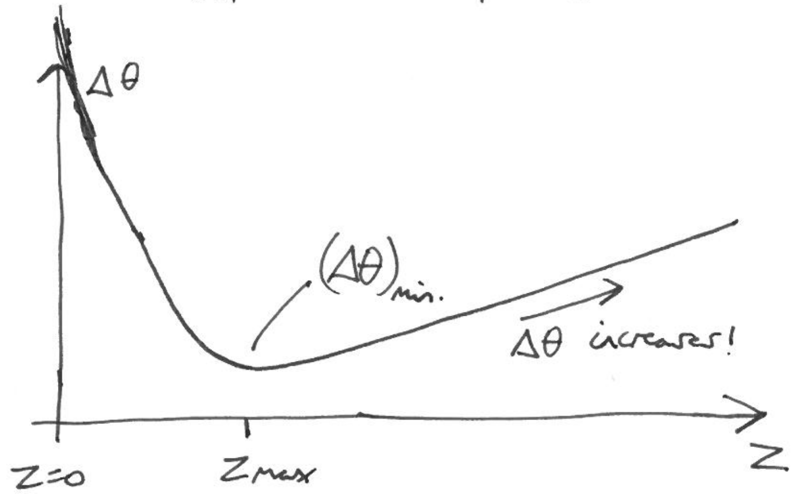
Sketch $d_A(z)$. We know that $d_A = 0$ at $z=0$, and we know it has a maximum at $z = z_{max} = 1.25$.



$$\Delta\theta \propto \frac{1}{d_A(z)}$$

(d is fixed). So, sketch the inverse of $d_A(z)$:

[3 MARKS] for sensible sketch



$\Delta\theta$ has divergence at $z=0$ (as expected, if we are very, very close to the object.

$\Delta\theta$ has <u>minimum</u> at $z = z_{max}$. Smallest possible angular size of this object

[2 MARKS]

$\Delta\theta$ <u>increases</u> at $z > z_{max}$.

The angular size gets bigger the further away the object is!

$\Rightarrow$ Objects at very high redshift appear <u>larger</u> on the sky than objects at intermediate redshifts!

[2 MARKS]

(This is a projection effect due to how light rays are 'bent' (their angles change) as they travel through a universe that is expanding.)