Week 6 Assessed Question

Spectral energy distribution:

$$u(\nu,T) = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/k_BT} - 1}$$

(a) Integrate over frequency (all frequencies!):

$$\rho_\gamma = \int_0^\infty u(\nu,T)\, d\nu. \qquad [2 \text{ MARKS}]$$

Substitution: $x = \frac{h\nu}{k_BT} \rightarrow \frac{dx}{d\nu} = \frac{h}{k_BT}$ and $\nu = \frac{k_BTx}{h}$.

$$\rho_\gamma = \int_0^\infty \frac{8\pi h}{c^3} \cdot \left(\frac{k_BT}{h}\right)^3 \cdot \frac{x^3}{e^x - 1} \cdot \left(\frac{k_BT}{h}\right) \cdot dx$$

same limits, since $x = \infty$ when $\nu = \infty$ and $x = 0$ when $\nu = 0$

[3 MARKS]

$$= \frac{8\pi h}{c^3} \cdot \left(\frac{k_BT}{h}\right)^4 \cdot \underbrace{\int_0^\infty \frac{x^3}{e^x-1}\, dx}_{=\frac{\pi^4}{15}}$$

$= \frac{\pi^4}{15}$ from standard integral.

$$\rightarrow \rho_\gamma = \frac{8\pi^5}{15} \cdot \frac{k_B^4}{c^3 h^3} \cdot T^4 \qquad [2 \text{ MARKS}]$$

(b) Calculate $\Omega_{CMB} = \dfrac{\rho_{CMB}(t_0)}{\rho_{cr,0}}$

$$\rho_{cr,0} = \frac{3H_0^2}{8\pi G} \quad \text{and} \quad H_0 = 67.4 \text{ km/s/Mpc}$$

$$= \underline{8.53 \times 10^{-27} \text{ kg m}^{-3}} \qquad [2 \text{ MARKS}]$$

$$\rho_{CMB}(t_0) = \rho_\gamma(T = 2.725 K)$$

$$= \frac{8\pi^5 k_B^4}{15 h^3 c^3} (2.725 K)^4 = \underline{4.172 \times 10^{-14} \text{ J m}^{-3}}.$$

[2 MARKS]

This is an energy density, but we calculated $\rho_{cr,0}$ as a mass density.

Conversion factor is c^2 (from $E = mc^2$)

$$\rightarrow \frac{\rho_{CMB}}{c^2} = \underline{4.642 \times 10^{-31} \text{ kg m}^{-3}} \qquad [1 \text{ MARK}]$$

$$\Omega_{CMB} = \frac{(\rho_{CMB}/c^2)}{\rho_{cr,0}} = \underline{5.44 \times 10^{-5}} \qquad [1 \text{ MARK}]$$

(c) $\Omega_M = 0.315$.

Calculate ρ_{CMB} / ρ_M at decoupling, $z_{dec} \simeq 1090$.

We know that $\rho_M(z) = \rho_{cr,0}\, \Omega_M \underbrace{(1+z)^3}_{= a^{-3}}$

and for CMB radiation we should find:

$$\rho_{CMB}(z) = \rho_{cr,0}\, \Omega_{CMB} \underbrace{(1+z)^4}_{= a^{-4}\text{, radiation.}}$$

$$\rightarrow \frac{\rho_{CMB}(z_{dec})}{\rho_M(z_{dec})} = \frac{\Omega_{CMB}}{\Omega_M} \times (1+z) \qquad [2 \text{ MARKS}]$$

$$= \frac{5.44 \times 10^{-5}}{0.315} \times 1091 = \underline{0.188} \qquad [2 \text{ MARKS}]$$

So CMB energy density has only $\sim \frac{1}{5}$ the matter density at decoupling.

(d) $z_{eq} = 3402$ in our Universe.

We know that $\Omega_{CMB} = 5.44 \times 10^{-5}$ and $\Omega_m = 0.315$

$\Omega_r = \Omega_{CMB} + \Omega_\nu$, both CMB + neutrinos treated as radiation.

We are given z_{eq}, redshift of matter-radiation equality,

$$\rho_m(z_{eq}) = \rho_r(z_{eq}).$$

$$\Rightarrow \Omega_m (1+z_{eq})^3 = \Omega_r (1+z_{eq})^4$$

So $\Omega_r = \dfrac{\Omega_m}{1+z_{eq}} = \underline{9.26 \times 10^{-5}}$ total fractional density of <u>all</u> radiation.

[2 MARKS]

$\Rightarrow \underbrace{\Omega_r = \Omega_{CMB} + \Omega_\nu}_{[1 \text{ MARK}]}$, so $\Omega_\nu = \Omega_r - \Omega_{CMB}$

$= (9.26 - 5.44) \times 10^{-5}$

$= \underline{3.82 \times 10^{-5}}$ [2 MARKS]

fractional neutrino density.

How to turn this into a temperature?

Well:

$$\left.\begin{aligned} \rho_{CMB} &\propto \Omega_{CMB}(1+z)^4 \propto T_{CMB}^4 \\ \rho_\nu &\propto \Omega_\nu (1+z)^4 \propto T_\nu^4 \end{aligned}\right\}$$

Proportionality constants are the same!

$$\Rightarrow \frac{\Omega_{CMB}\cancel{(1+z)^4}}{\Omega_\nu \cancel{(1+z)^4}} = \frac{T_{CMB}^4}{T_\nu^4} \qquad [2 \text{ MARKS}]$$

$$\text{so } T_\nu = T_{CMB} \times \left(\frac{\Omega_\nu}{\Omega_{CMB}}\right)^{\frac{1}{4}} = 2.725\,K \times \left(\frac{3.82}{5.44}\right)^{\frac{1}{4}}$$

$$= \underline{2.5\,K}$$

[1 MARK] $\longrightarrow$

neutrinos are slightly colder than photons

(N.B. A more correct calculation would take into account the fact that neutrinos are fermions and photons are bosons, so have slightly different no. of degrees of freedom. $T_\nu \simeq 1.95\,K$ in this case)