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Simple derivation of the Bohr–Wheeler spontaneous fission limit

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A simple model for establishing the Bohr and Wheeler limit for Z^2/A against spontaneous fission is developed. The physics involved is sufficiently simple to make the model suitable for sophomore students. Beyond showing how fundamental energy considerations lead to a limit for Z^2/A, the model helps to make clear the physical origin of the numerical values of the surface and Coulomb terms in the semiempirical mass formula. The resulting limit against spontaneous fission, $Z^2/A \sim 60$, is in fair agreement with the original Bohr and Wheeler value of ~ 48.

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I. INTRODUCTION

In their landmark paper on the mechanism of nuclear fission, Bohr and Wheeler[1] established a natural limit $Z^2/A \sim 48$ beyond which nuclei are unstable against spontaneous fission, where Z and A are the atomic and nucleon numbers, respectively. To establish this limit they modeled the shape of a ''liquid drop'' nucleus as a sum of Legendre polynomials configured to conserve volume as the nucleus deforms, and considered the total energy of the nucleus to be the sum of two contributions: a surface energy U_A proportional to the surface area of the nucleus, and an electrostatic contribution U_E. The surface energy term originates from the fact that nucleons near the surface of the nucleus are less strongly bound than those inside, and the Coulomb term arises from the mutual repulsion of the protons. If a nucleus is deformed from its original spherical shape, U_A will increase due to the consequent increase in surface area, while U_E decreases as the nuclear charge becomes more spread out. If $(U_A + U_E)_{\mathrm{deformed}} < (U_A + U_E)_{\mathrm{original}}$, the nucleus will be unstable against fission.

The detailed calculations carried out by Bohr and Wheeler are beyond the level of most undergraduate students. For this reason the usual textbook approach is to quote expressions for the surface and Coulomb energies of nuclei modeled as ellipsoids and then calculate the difference in energy between a spherical nucleus and an ellipsoid of the same volume.[2] Because of the fundamental importance of the concept of a limit on Z^2/A due to spontaneous fission, it is worthwhile to develop an approximate treatment suitable for presentation at a sophomore level that utilizes nuclei modeled as spheres. Such a derivation is the purpose of this paper.

II. FISSION MODEL

We begin with a spherical parent nucleus of radius R_0 as shown in Fig. 1(a). Imagine that this nucleus splits into two spherical product nuclei of radii R_1 and R_2 as shown in Fig. 1(b). If we assume that the density of nuclear matter is constant, conservation of nucleon number demands that the volume be conserved,

$$R_0^3 = R_1^3 + R_2^3. \tag{1}$$

We define the mass ratio of the fission products as

$$f = \frac{R_1^3}{R_2^3}. \tag{2}$$

This ratio could be defined as the inverse of that adopted here, a point to which we shall return. In terms of the mass ratio, the radii of the product nuclei are

$$R_1 = R_0 \left(\frac{f}{1+f} \right)^{1/3} \tag{3}$$

and

$$R_2 = R_0 \left(\frac{1}{1+f} \right)^{1/3}. \tag{4}$$

Following Bohr and Wheeler,[1] we take the energy of the system to comprise two contributions: a surface energy proportional to the surface area of the system, and the Coulomb term. Because students will normally have been exposed to the semiempirical mass formula of the liquid-drop model before fission is discussed, they should be comfortable with the meaning of these two energies. There is no need to incorporate a volume term because we assume that volume is conserved.

The surface energy of the original nucleus can be written as

$$U_A^{\mathrm{orig}} = a_S R_0^2, \tag{5}$$

where a_S is to be determined. That for the fissioned nucleus will be

$$U_A^{\mathrm{fiss}} = a_S (R_1^2 + R_2^2) = a_S R_0^2 \alpha, \tag{6}$$

where

$$\alpha = \frac{f^{2/3} + 1}{(1+f)^{2/3}}. \tag{7}$$

The Coulomb energy is slightly more involved. We begin with the result that the electrostatic energy of a sphere of charge of radius r is given by

$$U_E^{\mathrm{self}} = \left(\frac{4\pi\rho^2}{15\varepsilon_0} \right) r^5, \tag{8}$$

where ρ is the charge density. (This result can be derived by imagining a sphere of charge to be constructed by adding thin shells of charge to an existing core and integrating the electrostatic potential $Q_{\mathrm{shell}} Q_{\mathrm{core}} / 4\pi\varepsilon_0 r$ from $r = 0$ to the fi-

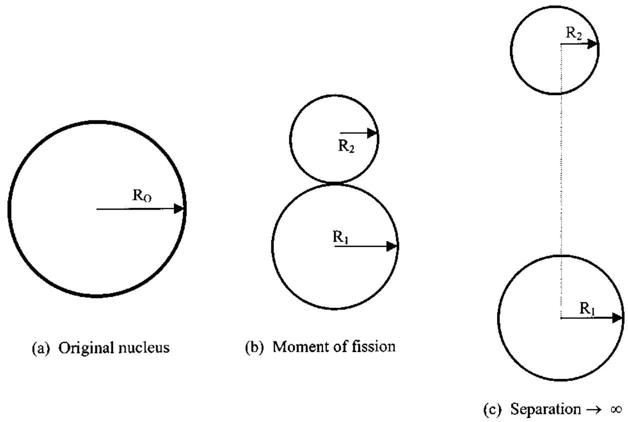

Fig. 1. Schematic illustration of the fission process: (a) an initially spherical nucleus divides into (b) two spherical product nuclei, which then (c) separate due to their mutual repulsion.

nal radius. U_E^{self} actually has an overall $1/r$ dependence because $\rho \propto r^{-3}$.) In the present case, $\rho = 3Ze/4\pi R_0^3$, where Z is the atomic number of the parent nucleus. This charge density gives $4\pi\rho^2/15\varepsilon_0 = 3Z^2e^2/20\pi\varepsilon_0 R_0^6$, and

$$U_E^{\mathrm{orig}} = \left(\frac{3e^2Z^2}{20\pi\varepsilon_0 R_0}\right). \tag{9}$$

The electrostatic energy of the system at the moment of fission [Fig. 1(b)] is the sum of each of the product nuclei plus the point-charge interaction between them

$$U_E^{\mathrm{fiss}} = \left(\frac{3e^2Z^2}{20\pi\varepsilon_0 R_0^6}\right)(R_1^5 + R_2^5) + \frac{Q_1Q_2}{4\pi\varepsilon_0(R_1+R_2)}, \tag{10}$$

where Q_1 and Q_2 are the charges of the product nuclei. In terms of the mass ratio of the fission products, U_E reduces to

$$U_E^{\mathrm{fiss}} = \left(\frac{3e^2Z^2}{20\pi\varepsilon_0 R_0}\right)(\beta+\gamma), \tag{11}$$

where

$$\beta = \frac{f^{5/3}+1}{(1+f)^{5/3}} \tag{12}$$

and

$$\gamma = \frac{(5/3)f}{(1+f)^{5/3}(f^{1/3}+1)}. \tag{13}$$

Next consider the common factor appearing in Eqs. (9) and (11). Empirically, nuclear radii behave as $R \sim aA^{1/3}$, where $a \sim 1.2$ fm. If we incorporate this approximation and substitute values for the constants, we obtain

$$\frac{3e^2Z^2}{20\pi\varepsilon_0 R_0} \sim 0.72\left(\frac{Z^2}{A^{1/3}}\right) \text{ MeV}, \tag{14}$$

which explains the origin of the numerical value of the coefficient of the Coulomb term in the semiempirical mass formula. If the same empirical radius–nucleon number model is incorporated into the surface-energy expressions, we can absorb the factor of 1.2 fm into the definition of a_S and write Eqs. (5) and (6) as $U_A^{\mathrm{orig}} = a_S A^{2/3}$ and $U_A^{\mathrm{fiss}} = a_S A^{2/3}\alpha$; the units of a_S will emerge as MeV.

Table I. Fission reactions and derived surface energy and spontaneous-fission limit parameters.

Fission products of ${}^1_0n + {}^{235}_{92}\mathrm{U}$	Q (MeV)	f	a_S (MeV)	(Z^2/A)
${}^{141}_{56}\mathrm{Ba} + {}^{92}_{36}\mathrm{Kr} + 3({}^1_0n)$	173.2	1.53	18.3	61.4
${}^{139}_{54}\mathrm{Xe} + {}^{95}_{38}\mathrm{Sr} + 2({}^1_0n)$	183.6	1.46	17.5	58.6
${}^{116}_{46}\mathrm{Pd} + {}^{116}_{46}\mathrm{Pd} + 4({}^1_0n)$	177.0	1.00	19.0	63.7
${}^{208}_{82}\mathrm{Pb} + {}^{26}_{10}\mathrm{Ne} + 2({}^1_0n)$	54.2	8.00	16.3	54.7

We can now consider the question of spontaneous fission. If the total energy of the two nuclei in the fission circumstance shown in Fig. 1(b) is *less* than that for the original nucleus of Fig. 1(a), then the system will fission. That is, spontaneous fission will occur if

$$U_A^{\mathrm{orig}} + U_E^{\mathrm{orig}} > U_A^{\mathrm{fiss}} + U_E^{\mathrm{fiss}}. \tag{15}$$

If we use Eqs. (5)–(7) (modified with $R_0 \propto A^{1/3}$) and Eqs. (9)–(14), this condition reduces to

$$\frac{Z^2}{A} > \frac{a_S(\alpha-1)}{0.72(1-\beta-\gamma)}. \tag{16}$$

Estimating the Z^2/A stability limit apparently demands selecting an appropriate mass ratio and knowing the value of a_S. For the latter, we could adopt a value from the semiempirical mass formula, but it is more satisfying to derive a value for a_S on direct physical grounds. This derivation is given in Sec. III, after which the question of an appropriate mass ratio is considered.

III. SURFACE ENERGY CALIBRATION

To estimate the value of a_S, we appeal to the fact that fission can be induced by slow neutrons with (for the case of uranium) $Q \sim 170$ MeV of energy being liberated. In our notation, this energy release can be expressed as

$$(U_A^{\mathrm{orig}} + U_E^{\mathrm{orig}}) - (U_A^{\infty} + U_E^{\infty}) = Q, \tag{17}$$

where U_A^{∞} and U_E^{∞}, respectively, designate the surface and electrical energies of the system when the product nuclei are infinitely far apart. Because the areas of the product nuclei do not change following fission, $U_A^{\infty} = U_A^{\mathrm{fiss}}$ [see Eqs. (6) and (7)]. U_E^{∞} is given by Eq. (10) without the point–charge interaction term, that is, Eq. (11) without the γ term. We substitute these equations and Eq. (14) into Eq. (17) and find

$$a_S = \frac{(Q/A^{2/3}) - 0.72(Z^2/A)(1-\beta)}{(1-\alpha)}, \tag{18}$$

where A and Z refer to the parent nucleus, *not* the general (Z^2/A) spontaneous-fission limit we seek.

The values of a_S derived in this way from a number of fission reactions are shown in Table I. The first reaction is representative of the Hahn and Strassmann fission-discovery reaction; the second is concocted to have the masses of the fission products to be 139 and 95, the values claimed by Weinberg and Wigner[3] to be the most probable mass yields in slow-neutron fission of ^{235}U. The mass ratios given in Table I are those of the fission products, neglecting any emitted neutrons. The last two reactions are less probable ones chosen to give a sense of how sensitive a_S is to the choice of calibrating reaction. As we might expect, a_S is in fact fairly *insensitive* to the choice of the calibrating reaction. The values of a_S derived here are consistent with those quoted in numerical fits of the semiempirical mass formula.[4]

IV. SPONTANEOUS FISSION LIMIT

With a sense of the magnitude of a_S in hand, we can proceed to estimate the stability limit for Z^2/A against spontaneous fission based on Eq. (16). If the limiting value of Z^2/A is to be a matter of fundamental physics, it should be independent of any fission-product mass ratio, a consideration that begs the question of what value of f to assume in α, β, and γ in Eq. (16). In particular, it would make no sense to use the mass ratio for the *induced* reaction used to calibrate a_S to determine a limit against *spontaneous* fission. To resolve this dilemma, recall that f could just as well have been defined as the inverse of what was adopted in Eq. (2). The only value of f that is in any sense unique is therefore $f=1$. Indeed, plots of the right-hand side of Eq. (16) versus f for fixed values of a_S reveal that a minimum always occurs at $f=1$, symmetric (in the sense of $f\rightarrow 1/f$) about $f=1$. To establish a lower limit to the spontaneous-fission condition, we take Eq. (16) evaluated at $f=1$,

$$\left(\frac{Z^2}{A}\right) \sim 3.356 a_S . \tag{19}$$

Limiting values of Z^2/A calculated from Eq. (19) are given in the last column of Table I. Although these values are somewhat high ($\sim$25%) compared to the Bohr and Wheeler value of 48, the agreement is reasonable given the simplicity of the present model.

Finally, we can roughly estimate the maximum value of Z beyond which nuclei will be unstable against spontaneous fission. From the data given in the online version of the January 2000 edition of the *Nuclear Wallet Cards*,[5] we find that there are 351 stable isotopes, that is, ones that occur naturally or that have half-lives >100 years. A plot of A versus Z for these isotopes can be approximately fit by a power law,

$$A \sim 1.686 Z^{1.087}, \tag{20}$$

with a correlation coefficient r^2 of 0.997. This fit actually slightly underestimates $A(Z)$ for heavy nuclei, giving $A \sim 230$ for $Z=92$, but is sufficient for our purposes. For a limiting Z^2/A of 60, this fit predicts a maximum stable Z of about 157; the Bohr and Wheeler value of 48 gives a maximum Z of about 123.

V. CONCLUSION

A simple model for establishing the Bohr and Wheeler limit for Z^2/A against spontaneous fission has been developed. Because this model incorporates only spherical geometry, concepts of surface and electrostatic interaction energies from the semiempirical mass formula, nuclear radii as a function of nucleon number and energy release in fission reactions, it is suitable for presentation at the sophomore level. A bonus of this approach is that it helps to make clear the physical origin of the numerical values of the surface and Coulomb terms in the semiempirical mass formula. The resulting limit against spontaneous fission, $Z^2/A \sim 60$, is in reasonable agreement with more sophisticated calculations that treat a fissioning nucleus as a deforming ellipsoid in contrast to the two spheres assumed here.

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[4]See, for example, R. A. Serway, C. J. Moses, and C. A. Moyer, *Modern Physics*, 2nd ed. (Harcourt College, Fort Worth, TX, 1997), p. 542.
[5]J. K. Tuli, *Nuclear Wallet Cards*, 6th ed. (National Nuclear Data Center, Brookhaven National Laboratory). Available online at ⟨www.nndc.bnl.gov/wallet⟩.