Week 6: Cosmic Microwave Background radiation

Please hand in the completed problems by **Wednesday 13th of November at 4pm**. *Please show your working and write neatly, staple all sheets together, and write your name and student number at the top of the first sheet.*

1. **Maths practice: Change of variables**

   (a) By writing $x = \nu/\nu_0$, convert the integral $I = \int \nu^\beta d\nu$ into a dimensionless integral times a dimensionful prefactor. What are the dimensions of the prefactor?

   (b) The first few Legendre polynomials can be written as

   $$\mathcal{P}_0(\theta) = 1; \quad \mathcal{P}_1(\theta) = \cos\theta; \quad \mathcal{P}_2(\theta) = \frac{1}{2}\left(3\cos^2\theta - 1\right).$$

   By performing the substitution $\mu = \cos\theta$ and then integrating, show that these polynomials satisfy an *orthogonality relation*,

   $$\int_0^\pi \mathcal{P}_n(\theta)\mathcal{P}_m(\theta)\sin\theta d\theta = 0 \quad \text{when}\, n \neq m.$$

   (c) By using the change of variables $x = r\cos\theta$ and $y = r\sin\theta$, solve the integral

   $$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx\, dy.$$

   Hint: Calculate the *Jacobian* to do the change of variables, and then use a further substitution, $u = -r^2$, to simplify the integral that you get.

2. **Redshift of decoupling**
   Photons decouple from baryons when their mean free path becomes larger than the Hubble radius.

   (a) Write down expressions for (i) the comoving Hubble radius, $r_{\rm HR}$, in a flat matter-only Universe, and (ii) the *comoving* mean free path of photons.

   (b) Show that the number density of electrons in a fully-ionised gas scales with the scale factor like $n_e \propto a^{-3}$.

   (c) Decoupling occurs in the matter-dominated era, when the expansion rate $H(a)$ can be approximated using the flat, matter-only Friedmann equation.
   By substituting appropriate expressions for $H(a)$ and $n_e(a)$ into your answer for part (a) above, derive an expression for the redshift at which decoupling occurred.

3. **Formation of the CMB**
   Briefly explain each of the following terms:

   (i) Recombination; (ii) Surface of last scattering; (iii) Decoupling.

4. **Lookback time**
   Consider a flat, matter-only universe with $H_0 = 70$ km/s/Mpc, where decoupling happened at $z_{\rm dec} \approx 1090$.

   (a) By solving the Friedmann equation, find an expression for the time interval between two different redshifts, $\Delta t = t(z_1) - t(z_2)$, in this universe.

   (b) What is the time interval between when decoupling happened and today in this universe?

   (c) Now consider a galaxy observed at redshift $z = 2$. How long ago was the light from that galaxy emitted?

   (d) What was the temperature of the CMB when the light from the galaxy was emitted? Assume the CMB temperature today is $T_0 = 2.7$ K.

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5. **Distance to the surface of last-scattering**
   Consider a flat, matter-only Universe with $H_0 = 70$ km/s/Mpc.

   (a) Derive an expression for the comoving distance travelled by light that was emitted at a scale factor a that reaches us today at $a = 1$.

   (b) Calculate the comoving distance to the surface of last scattering if decoupling occurred at $z_{\rm dec} = 1090$.

   (c) Calculate the angular diameter distance, d_A, and luminosity distance, d_L, to the surface of last scattering.

   (d) Is the surface of last scattering within the comoving Hubble radius today, $r_{\rm HR}(a = 1)$?

6. **Number density of CMB photons**
   The number density of photons in a blackbody radiation field is given by

$$n_\gamma = \frac{8\pi}{(hc)^3} \int_0^\infty \frac{E^2 dE}{\exp(E/k_B T) - 1},$$

   where E is the photon energy and T is the blackbody temperature.

   (a) Use the standard integral $\int_0^\infty x^2 (e^x - 1)^{-1} dx = 2.404$ to simplify the expression for n_γ.

   (b) Calculate n_γ for the CMB observed today, at a temperature of $T \approx 2.7$ K.

Assessed question: Energy density of CMB radiation

The spectral energy density (energy per unit volume per unit frequency) of a thermal (blackbody) gas of photons at temperature T is given by

$$u(\nu, T) = \frac{8\pi h}{c^3} \frac{\nu^3}{\exp\left(\frac{h\nu}{k_{\rm B}T}\right) - 1}.$$

(a) By integrating $u(\nu, T)$ over frequency ν, show that the total energy density of a thermal photon gas is

$$\rho_\gamma(T) = \frac{8\pi^5 k_{\rm B}^4}{15 h^3 c^3} T^4.$$

Hint: Perform a change of variables and then use the standard integral $\int_0^\infty x^3 (e^x - 1)^{-1} dx = \pi^4/15$.
[7 marks]

(b) Calculate the fractional energy density of CMB radiation today, $\Omega_{\rm CMB} \equiv \rho_{\rm CMB}(t_0)/\rho_{\rm crit,0}$, if its blackbody temperature is $T = 2.725$ K and the value of the Hubble parameter is $H_0 = 67.4$ km/s/Mpc.

Hint: To convert between a mass density and energy density, recall that $E = mc^2$. The correct order of magnitude of the final result is $\Omega_{\rm CMB} \sim 10^{-5}$.
[6 marks]

(c) The fractional energy density of matter today is measured to be $\Omega_{\rm m} = 0.315$. Calculate the ratio of the energy densities of CMB photons and matter, $\rho_{\rm CMB}(z)/\rho_m(z)$, at decoupling ($z_{\rm dec} \approx 1090$).
[4 marks]

(d) The redshift of matter-radiation equality is measured to be $z_{\rm eq} = 3402$ in our Universe.

Assume that the only two types of radiation are CMB photons with $T = 2.725$ K and a thermal gas of (massless) neutrinos with temperature $T = T_\nu$. Use the measured value $z_{\rm eq} = 3402$ and the result from part (a) to calculate a blackbody temperature T_ν for the neutrinos.
[8 marks]