Assessed question – Week 3 Physical Cosmology

(a) Friedmann Equation:

$$H^2(a) = \left(\frac{1}{a}\frac{da}{dt}\right)^2 = H_0^2\left(\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_k a^{-2} + \Omega_\Lambda\right)$$

[4 MARKS]

(b) Age for matter-dominated from lecture notes:

$$t_0 = \frac{2}{3H_0}$$ [2 MARKS]

Radiation-dominated: $t_0 = \int_0^1 \frac{da}{aH}$ and $H = H_0 a^{-2}$ for $\Omega_r = 1$.

$$= \int_0^1 \frac{da}{H_0 \cdot a \cdot a^{-2}}$$

$$= \frac{1}{H_0}\int_0^1 a\, da = \frac{1}{H_0}\left[\frac{a^2}{2}\right]_0^1$$

$$= \frac{1}{2H_0}$$ [2 MARKS]

$\Rightarrow$ $t_{0,rad} < t_{0,matter}$

Without solving for t_0, we can see that the 99% matter universe will be an intermediate case. Behaves like matter at large t, radiation at small t.

$\longrightarrow$ In increasing order of age: Radiation only (youngest)
99% matter, 1% radiation
Matter only (oldest)

[2 MARKS]

(c) Matter-radiation equality when $\rho_r(t) = \rho_m(t)$

$$\Rightarrow \quad \Omega_r a^{-4} = \Omega_m a^{-3} \qquad \text{when } a = a_{eq}$$

(scale factor of equality)

$$\rightarrow \quad \Omega_r a_{eq}^{-4} = \Omega_m a_{eq}^{-3}$$

$$\text{so} \quad a_{eq} = \frac{\Omega_r}{\Omega_m}$$

Redshift $(1+z_{eq}) = a_{eq}^{-1} \rightarrow z_{eq} = \frac{1}{a_{eq}} - 1$

$$= \frac{\Omega_m}{\Omega_r} - 1$$

In this universe, 99% matter and 1% radiation <u>today</u>

[3 MARKS] for correct expression

so $\Omega_m = 0.99$, $\Omega_r = 0.01$

$$\Rightarrow z_{eq} = \frac{0.99}{0.01} - 1 = 98$$

[1 MARK] for correct value

(d) Deceleration parameter, $q(a) = -\left(1 + \dot{H}/H^2\right)$

$\dot{H} = \frac{dH}{dt}$. In this universe:

$$H^2 = H_0^2\left(\Omega_m a^{-3} + \Omega_r a^{-4}\right)$$

[2 MARKS]

$\frac{dH}{dt}$; can get this by differentiating Friedmann Eqn. directly:

$$\frac{d}{dt}(H^2) = 2H\dot{H} = H_0^2\left(\Omega_m a^{-4}.(-3\dot{a}) + \Omega_r a^{-5}(-4\dot{a})\right)$$

(d) cont'd

$$\rightarrow 2H\dot{H} = -H_0^2 \frac{\dot{a}}{a}\left(3\Omega_m a^{-3} + 4\Omega_r a^{-4}\right)$$

taken a factor of a^{-1} outside brackets

Rearrange: $$\dot{H} = -\frac{H_0^2 \cdot \cancel{H}}{2\cancel{H}}\left(3\Omega_m a^{-3} + 4\Omega_r a^{-4}\right)$$

[2 MARKS]

Divide by H^2 and add 1:

$$-q(a) = 1 + \frac{\dot{H}}{H^2} = -\frac{\cancel{H_0^2}}{2}\frac{\left(3\Omega_m a^{-3} + 4\Omega_r a^{-4}\right)}{\cancel{H_0^2}\left(\Omega_m a^{-3} + \Omega_r a^{-4}\right)} + 1$$

Multiply 1 by $\frac{2H^2}{2H^2}$:

$$-q(a) = -\frac{\left(3\Omega_m a^{-3} + 4\Omega_r a^{-4}\right)}{2\left(\Omega_m a^{-3} + \Omega_r a^{-4}\right)} + \frac{2\left(\Omega_m a^{-3} + \Omega_r a^{-4}\right)}{2\left(\Omega_m a^{-3} + \Omega_r a^{-4}\right)}$$

simplify:

$$-q(a) = \left[2\left(\Omega_m a^{-3} + \Omega_r a^{-4}\right)\right]^{-1} \times \left(-\Omega_m a^{-3} - 2\Omega_r a^{-4}\right)$$

$$\rightarrow q(a) = \frac{\Omega_m a + 2\Omega_r}{2\Omega_m a + 2\Omega_r}$$

[2 MARKS] for similar expression

[1 MARK] for doing a good job of simplifying it

(e) Use $q(a) = \dfrac{\Omega_m a + 2\Omega_r}{2\Omega_m a + 2\Omega_r}$

Matter-only: $\Omega_m = 1$, $\Omega_r = 0$

$t = t_0$, so $a = 1$

$\rightarrow \underline{q(t_0) = \frac{1}{2}}$ [2 MARKS]

Radiation-only: ~~$\Omega_m = 0$~~ $\Omega_m = 0$, $\Omega_r = 1$

$\rightarrow \underline{q(t_0) = 1}$ [2 MARKS]