Week 4: Distances and horizons

Please hand in the completed problems by **Wednesday 23rd of October at 4pm**. *Please show your working and write neatly, staple all sheets together, and write your name and student number at the top of the first sheet.*

1. **Maths practice: Maxima and minima of functions**
   (a) Find the extremum (max. or min. value) of the function $y = x^2$. Is this a maximum or a minimum?
   (b) Find the extrema of the function $y = 4x^2 + 2x - 3$.
   (c) Find the extrema of the function $y = x^2 + 3e^{-x^2}$.

2. **Etherington distance-duality relation**
   The Etherington distance-duality relation is the statement that $d_L(z) = (1+z)^2 d_A(z)$.
   (a) Calculate the Taylor expansion of the luminosity distance, $d_L(z)$, up to second order about $z = 0$. Assume an arbitrary expansion rate $H(z)$ (i.e. don't substitute in the Friedmann equation for $H(z)$).
   (b) Derive a similar Taylor expansion for the angular diameter distance, $d_A(z)$.
   (c) Compare the first-order parts of the two Taylor expansions. What do you notice?

3. **Scale factor in a closed universe**
   The scale factor in a closed universe, containing only matter and positive curvature, is given by the following parametric solution:

$$
\begin{aligned}
a(\tau) &\propto (1 - \cos\tau) \\
t(\tau) &\propto (\tau - \sin\tau),
\end{aligned}
$$

   where τ is the conformal time.
   (a) Show that this is indeed a solution to the Friedmann equation in a closed universe with matter.
   (b) Derive the constants of proportionality for $a(\tau)$ and $t(\tau)$ in terms of H_0, Ω_m, and Ω_k.

4. **Horizons in a matter-only universe**
   Consider a flat, matter-only universe, with a scale factor that evolves with time as $a(t) = (3H_0 t/2)^{\frac{2}{3}}$.
   (a) Derive an expression for the Hubble horizon, $r_{\rm HR}(a)$, in this universe.
   (b) Derive an expression for the particle horizon, $r_{\rm H}(a)$, in this universe.
   (c) Sketch a graph showing how $r_{\rm HR}$ and $r_{\rm H}$ depend on scale factor.

Assessed question: Angular diameter distance

Consider a flat, matter-only universe.

(a) Write down the definition of the angular diameter distance, $d_A(z)$, in terms of the angular size $\Delta\theta$ and physical size d of a distant object observed at redshift z. How is the angular diameter distance related to the comoving distance to the object?
[4 marks]

(b) Derive an expression for $d_A(z)$, the angular diameter distance as a function of redshift, in this universe.
[7 marks]

(c) Calculate the redshift at which the angular diameter distance reaches its maximum value.
[7 marks]

(d) Consider an object of fixed physical size d. Explain how its angular size, $\Delta\theta$, would change depending on what redshift it was observed at in this universe (hint: sketch a graph). What are the implications for the angular sizes of objects that we see at great distances?
[7 marks]