Week 2: Geometry and distance

Please hand in the completed problems by **Wednesday 9th of October at 4pm**. *Please show your working and write neatly, staple all sheets together, and write your name and student number at the top of the first sheet.*

1. **Maths practice: Taylor expansions and trigonometry**

   (a) Consider a right-angled triangle with hypotenuse of length 20 and opposite side of length 4. (i) Find the length of the adjacent side. (ii) Find the opposite angle, in degrees.

   (b) Expand $y = (1+2x)^{-2}$ to linear order about $x = 0$.

   (c) Expand $y = \ln(1+x)$ to linear order about $x = 1$.

   (d) Expand $y = \sin(3x)$ to quadratic order about $x = 0$.

2. **Small-angle approximation**
   In cosmology, we are often dealing with small angles, $\theta \simeq 0$. This allows us to use the *small-angle approximation* to simplify some expressions.

   (a) In the small angle approximation, show that $\tan\theta \simeq \theta$.

   (b) By evaluating the next-to-leading-order term in the Taylor expansion, find the angle at which the small angle approximation to $\tan\theta$ differs from the exact solution by (approximately) more than 10%.

3. **Curved universe**
   Consider an ant walking around on the surface of a beach ball.

   (a) Using this as an analogy, explain what is meant by a spatially-closed universe.

   (b) What would be equivalent analogies for spatially flat and spatially open universes?

4. **Metric tensor**
   Consider a space-time with the following metric tensor, for coordinates (t, x, y, z):

$$g_{ab} = \begin{pmatrix} -c^2 & & & \\ & a_{\parallel}^2(t) & & \\ & & a_{\perp}^2(t) & \\ & & & a_{\perp}^2(t) \end{pmatrix}.$$

   (a) Write down the line element, ds^2, for this space-time.

   (b) What conditions must be satisfied for this to be an FLRW metric? (Hint: Consider the normalisation and rate of change of the two different scale factors, $a_{\parallel}$ and $a_{\perp}$.)

5. **Conformal factor**
   Two metrics, g and $\tilde{g}$, are said to be conformally equivalent if they satisfy a relation of the form $\tilde{g}_{ij} = f(\vec{x}, t) g_{ij}$, where $f(\vec{x}, t)$ is the *conformal factor*. Note how f is a scalar quantity, i.e. it is the same for all i and j.

   (a) The Minkowski metric of Special Relativity is given by $\eta_{ij} = \mathrm{diag}(-c^2, 1, 1, 1)$. By first performing an appropriate coordinate transformation, show that any flat FLRW metric is conformal to Minkowski.

   (b) Conformal transformations preserve angles (so angles in a spacetime described by metric g remain the same after transforming to metric $\tilde{g}$). Use this fact to explain why a *closed* FLRW spacetime can't be conformally equivalent to Minkowski.

(PLEASE TURN OVER)

Assessed question: Measuring the Hubble constant

Doubly-ionised Oxygen (O[III]) has an emission line with a rest-frame wavelength of 166.59 nm. The table below shows the measured wavelengths of the O[III] line from several galaxies, along with their parallax in micro-arcseconds. (We will assume that measurement errors are negligible.)

Galaxy	$\lambda_{\rm obs}^{\rm O[III]}$ (nm)	Parallax (μas)
A	167.090	0.0634
B	167.699	0.0369
C	168.472	0.0221
D	167.562	0.0462

(a) Calculate the redshift and 'recession velocity' of each galaxy.
[5 marks]

(b) Calculate the parallax distance to each galaxy (in pc).
[5 marks]

(c) Plot these quantities on a graph (draw it by hand). Add a straight line that fits the data points as well as possible. Measure its gradient and intercept.
[5 marks]

(d) Using this plot, infer the value of the Hubble parameter, H_0, in units of km/s/Mpc.
[5 marks]

(e) Do the points on the graph follow a perfectly straight line? If not, explain why.
[5 marks]