4.1 (c) From the definitions,

$$\frac{T_{relax}}{T_{cross}} = \frac{\dfrac{1}{6N \ln\left(\frac{b_{max}}{b_{min}}\right)} \cdot \dfrac{(Rv)^3}{(Gm)^2}}{\dfrac{2R}{v}}$$

Choose $b_{max} = R$, the size of the system

$b_{min} = \dfrac{2Gm}{v^2}$, strong encounter radius

(since $\frac{1}{2}v^2 = \frac{Gm}{r}$)

$$\therefore \frac{T_{relax}}{T_{cross}} = \frac{\dfrac{1}{6N \ln\left(\frac{Rv^2}{2Gm}\right)} \cdot \dfrac{(Rv)^3}{(Gm)^2}}{\dfrac{2R}{v}}$$

Also, assuming $v \sim \sqrt{\dfrac{GNm}{R}}$, then $\dfrac{Rv^2}{Gm} = N$

$$\therefore \frac{T_{relax}}{T_{cross}} = \frac{1}{12N \ln\left(\frac{Rv^2}{2Gm}\right)} \cdot \frac{R^2 v^4}{(Gm)^2}$$

$$= \frac{1}{12N \ln\left(\frac{N}{2}\right)} \cdot N^2$$

$$\therefore \underline{\frac{T_{relax}}{T_{cross}} \sim \frac{N}{12 \ln N}}$$

(since $\ln\left(\frac{N}{2}\right) \simeq \ln N$ for large N).

Solution 4.1 (d)

Assume that the stars are distributed with uniform density across the cluster and that it is spherical with a radius $R = 70$ pc $= 70 \times 3.0857 \times 10^{16}$ m $= 2.2 \times 10^{18}$ m.

The typical velocity of the stars is $100\ \mathrm{km\,s^{-1}} = 10^5\ \mathrm{m\,s^{-1}}$. The relaxation time is given by

$$T_{relax} \simeq \frac{1}{6N \ln\left(\frac{b_{max}}{b_{min}}\right)} \frac{(Rv)^3}{(Gm)^2} \ ,$$

for a uniform spherical system of radius R containing N stars of mass m moving with velocity v, where b_{max} and b_{min} are the maximum and minimum values of the impact parameter for stellar encounters and G is the constant of gravitation.

To proceed, make the approximation that all stars are moving with the typical velocity ($v = 10^5\ \mathrm{m\,s^{-1}}$), that all stars have a mass of one solar mass ($m = 1M_\odot = 1.989 \times 10^{30}$ kg), and that the minimum impact parameter is the strong encounter radius, $r_S = 2Gm/v^2$. So,

$$\begin{aligned}
T_{relax} &\simeq \frac{1}{6N \ln\left(\frac{Rv^2}{2Gm}\right)} \frac{(Rv)^3}{(Gm)^2} \\
&\simeq \frac{1}{6 \times 10^7 \ln\left(\frac{2.2\times10^{18}\times(10^5)^2}{2\times6.673\times10^{-11}\times1.989\times10^{30}}\right)} \frac{(2.2 \times 10^{18} \times 10^5)^3}{(6.673 \times 10^{-11} \times 1.989 \times 10^{30})^2} \ \mathrm{s} \\
&\simeq \frac{1}{6 \times 10^7 \ln(8.29 \times 10^6)} \frac{1.06 \times 10^{70}}{(1.76 \times 10^{40}} \ \mathrm{s} \ = \ 5.50 \times 10^{20}\ \mathrm{s} \\
&\simeq 5.50 \times 10^{20}/3.1557 \times 10^7\ \mathrm{yr} \ = \ 1.7 \times 10^{13}\ \mathrm{yr}
\end{aligned}$$

The relaxation time for this cluster around the nucleus is much longer than the likely age of the galaxy (age of galaxy $\simeq$ age of Universe $= 13.7 \times 10^9$ yr). So the dynamics of the stars can be modelled as a collisionless system over the lifetime of the galaxy.

The stars away from the nuclear region of the galaxy are also collisionless: in this case, the stars around the nucleus behave in the same way as the general stars of the galaxy as far as dynamical collisions are concerned.

[It should be noted that the densities of stars around the nuclei of many galaxies are so large that their dynamics are collisional: the relaxation times are small compared to the age of the galaxies. The case given in this problem is an exception.]

4.2 (a)

Virial theorem is $2\langle T\rangle + \langle V\rangle = 0$

where $\langle T\rangle$ and $\langle V\rangle$ are the time-averaged total KE and PE respectively.

$$T = \sum_{i=1}^{N} \frac{1}{2} m v^2 = \frac{1}{2} N m v^2 \quad , \text{ for N stars}$$

and averaging over time,

$$\langle T\rangle = \frac{1}{2} N m v^2$$

Given $V = -\frac{3}{5}\frac{GM^2}{R}$, then $\langle V\rangle = -\frac{3}{5}\frac{GM^2}{R}$

where M is the total mass, and substituting into the virial theorem,

$$2\left(\frac{1}{2} N m v^2\right) - \frac{3}{5}\frac{GM^2}{R} = 0$$

But total mass, $M = Nm$

$$\therefore \quad M v^2 = +\frac{3}{5}\frac{GM^2}{R} \qquad \therefore \quad v^2 = \frac{3}{5}\frac{GM}{R}$$

(b) In this case, the system is still evolving dynamically, so the virial theorem does not apply (the moment of inertia is still changing).

(c) In the case of hot gas in clusters of galaxies, the particles will be the ions in the gas.

4.2 (d)

Given the total internal P.E. is

$$V = -\frac{3\pi}{32}\frac{GM_{tot}^2}{a} \doteq \langle V \rangle$$

and since the time averaged KE must be

$$\langle T \rangle = \frac{1}{2} M_{tot} v^2$$

Applying the virial theorem

$$2\langle T \rangle + \langle V \rangle = 0$$

$$\therefore \quad 2\left(\frac{1}{2} M_{tot} v^2\right) - \frac{3\pi}{32}\frac{GM_{tot}^2}{a} = 0$$

$$\therefore \quad M_{tot} = \frac{32 a v^2}{3\pi G} = \frac{32 \times 2 \times 10^{17} \times (10^4)^2}{3\pi \times 6.67 \times 10^{-11}}$$

$$= 1.0 \times 10^{36} \text{ kg}$$

$$\sim 5.1 \times 10^5 M_\odot$$