SPA 7010 Problems 2

Problem 2.1

The virial theorem states that

$$2\langle T\rangle + \langle U\rangle = 0$$

where T is the total K.E. of the system, U is the total P.E. and $\langle\ \rangle$ denotes the time average.

Conditions for application:

isolated gravitating system

'relaxed' into dynamical equilibrium

non-rotating

Let the total mass of the cluster be M, radius R, velocity dispersion, σ.

$$\therefore \langle T\rangle \sim \frac{1}{2}M\sigma^2 \quad \text{and} \quad \langle U\rangle \sim -\frac{3GM^2}{5R}$$

So, using the virial theorem,

$$2\left(\frac{1}{2}M\sigma^2\right) - \frac{3GM^2}{5R} = 0 \qquad \therefore M = \frac{5R\sigma^2}{3G}$$

and given $R = 750\ \text{kpc} = 750 \times 3.09 \times 10^{19}\ \text{m}$

$\sigma = 800\ \text{kms}^{-1} = 8 \times 10^5\ \text{ms}^{-1}$

$$M = \frac{5 \times (750 \times 3.09 \times 10^{19}) \times (8 \times 10^5)^2}{3 \times 6.67 \times 10^{-11}}$$

$$= \frac{7.416 \times 10^{34}}{2.001 \times 10^{-10}} = 3.706 \times 10^{44}\ \text{kg} = \underline{1.8531 \times 10^{14}\ M_\odot}$$

Problem 2.2

Average K.E. $\langle T \rangle = \frac{1}{2} m V^2$

Average P.E. $\langle U \rangle = -\frac{GmM}{R}$

Virial theorem states $2\langle T \rangle + \langle U \rangle = 0$

$\therefore 2\left(\frac{1}{2} m V^2\right) - \frac{GmM}{R} = 0 \qquad \therefore V^2 = \frac{GM}{R}$

Since the orbit is circular,

$V = \frac{2\pi R}{T}$ where T is the orbital period

$\therefore \frac{4\pi^2 R^2}{T^2} = \frac{GM}{R} \qquad \therefore T^2 = \frac{4\pi^2 R^3}{GM}$

ie. $\underline{T^2 \propto R^3}$

This is Kepler's Third Law