MSci/MSc Examination

Main examination period 2017

SPA7018U-P Relativistic Waves and Quantum Fields
Duration: 2 hours 30 minutes

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Instructions:

Answer ONLY THREE questions. Each question carries 20 marks.

If you answer more questions than specified, only the first answers (up to the specified number) will be marked. Cross out any answers that you do not wish to be marked.

Only non-programmable calculators are permitted in this examination. Please state on your answer book the name and type of machine used.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Answer ONLY THREE of the five questions (a FORMULA SHEET is provided at the end of the paper)

Question 1: The Lorentz group

(a) A Lorentz transformation $\Lambda$ acts on spacetime coordinates $x^\mu$ as $x'^\mu = \Lambda^\mu_\nu x^\nu$, where $\Lambda$ satisfies $\Lambda^\mu_\rho \eta_{\mu\nu} \Lambda^\nu_\sigma = \eta_{\rho\sigma}$. Use this equation to show that the component $\Lambda^0_0$ of a Lorentz transformation satisfies the condition $|\Lambda^0_0| \geq 1$.

[4 marks]

(b) Use the relation $\Lambda^\mu_\rho \eta_{\mu\nu} \Lambda^\nu_\sigma = \eta_{\rho\sigma}$ to determine the number of independent parameters of the Lorentz group. [Hint: for this part of the question you may find it useful to consider infinitesimal Lorentz transformations: $\Lambda^\mu_\nu \simeq \delta^\mu_\nu + \alpha^\mu_\nu + O(\alpha^2)$.] 

[5 marks]

(c) A boost along the $x$-direction can be represented by the matrix

$$L(\theta) := \begin{pmatrix} \cosh \theta & \sinh \theta & 0 & 0 \\ \sinh \theta & \cosh \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\theta$ is called the rapidity of the boost. Write down the transformation $L(\theta_1) \cdot L(\theta_2)$ obtained by performing two consecutive boosts along the $x$-axis with rapidities $\theta_2$ and $\theta_1$. Explain why $L(\theta_1) \cdot L(\theta_2)$ is again a Lorentz transformation. Show that the rapidity of $L(\theta_1) \cdot L(\theta_2)$ is equal to $\theta_1 + \theta_2$, i.e. show that $L(\theta_1) \cdot L(\theta_2) = L(\theta_1 + \theta_2)$.

[5 marks]

(d) The coupling of a Dirac field $\psi$ to the electromagnetic field is realised through an interaction Lagrangian

$$\mathcal{L}_{\text{int}} = -e(\bar{\psi}\gamma^\mu \psi)A_\mu,$$

where $\gamma^\mu$ are the four Dirac matrices and $e$ is the electron charge. $A^\mu$ is the vector potential, which under a Lorentz transformation transforms as a four-vector, and $\bar{\psi} = \psi^\dagger \gamma^0$. Show that $\mathcal{L}_{\text{int}}$ is Lorentz invariant.

In this question you may use without proof that, under a Lorentz transformation $\Lambda$, the Dirac field $\psi(x)$ transforms as

$$\psi(x) \rightarrow \psi'(x') := S(\Lambda)\psi(x),$$

where the $4 \times 4$ matrix $S(\Lambda)$ satisfies $S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu_\nu \gamma^\nu$.

[6 marks]
Question 2: Quantisation of the complex scalar field

(a) The Lagrangian density for a complex scalar field $\phi$ is
\[ L = (\partial^\mu \phi^\dagger)(\partial_\mu \phi) - m^2 \phi^\dagger \phi. \]

Write down the Euler-Lagrange equations for the fields. Check explicitly that the following free-field expression for $\phi$:
\[ \phi(x) = \int d^3 p N(p) \left[ a(p) e^{-iE(p)t + i p \cdot x} + b^\dagger(p) e^{iE(p)t - i p \cdot x} \right], \]
satisfies the equations of motion, thereby determining $E(p)$ as a function of $p$. Here $N(p)$ is a normalisation factor.

[4 marks]

(b) The Hamiltonian of the theory, written in terms of oscillators, is given by
\[ H = \int d^3 p N(p) E(p) \left[ a^\dagger(p)a(p) + b(p)b^\dagger(p) \right]. \]

Write down the expression for the normal ordered Hamiltonian $H_{\text{norm ord}}$, and explain how it is obtained from the expression of $H$. Hence determine the expectation value $\langle 0 | H_{\text{norm ord}} | 0 \rangle$.

[4 marks]

(c) Use your expression for $H_{\text{norm ord}}$ to compute the energy of the states $a^\dagger(k)|0\rangle$ and $b^\dagger(k_1)a^\dagger(k_2)|0\rangle$, i.e. compute the quantities $H_{\text{norm ord}}(a^\dagger(k)|0\rangle)$ and $H_{\text{norm ord}}(b^\dagger(k_1)a^\dagger(k_2)|0\rangle)$.

[6 marks]

(d) The Noether charge associated to $U(1)$ phase transformations, written in terms of oscillators, has the form
\[ Q_{\text{norm ord}} = \int d^3 p N(p) \left[ a^\dagger(p)a(p) - b^\dagger(p)b(p) \right]. \]

Use this expression to find the $U(1)$ charge of the states $a^\dagger(k)|0\rangle$ and $b^\dagger(k_1)a^\dagger(k_2)|0\rangle$, and explain the physical interpretation of these two states. Finally, use the above expression for $Q_{\text{norm ord}}$ and that for the Hamiltonian $H_{\text{norm ord}}$ given in part (b) to prove that $[Q_{\text{norm ord}}, H_{\text{norm ord}}] = 0$.

[6 marks]

In this problem you may use without proof the commutation relations between the oscillators
\[ [a(p) , a^\dagger(q)] = [b(p) , b^\dagger(q)] = (2\pi)^3 2E(p) \delta^{(3)}(p - q), \]
\[ [a(p) , a(q)] = [b(p) , b(q)] = [a(p) , b(q)] = 0, \]
which are consistent with the choice of normalisation $N(p) := 1/[(2\pi)^3 2E(p)]$. Turn over
**Question 3: Quantisation of the Dirac field**

(a) Consider the two Lagrangian densities \( \mathcal{L} \) and \( \mathcal{L}' \) describing the dynamics of a free massive Dirac fermion,

\[
\mathcal{L} = \bar{\psi} \left( i \gamma^\mu \partial_\mu - m \right) \psi, \quad \mathcal{L}' = \frac{i}{2} \left( \bar{\psi} \gamma^\mu \partial_\mu \psi - \left( \partial_\mu \bar{\psi} \right) \gamma^\mu \psi \right) - m \bar{\psi} \psi,
\]

where \( \bar{\psi} := \psi^\dagger \gamma^0 \). For both Lagrangians \( \mathcal{L} \) and \( \mathcal{L}' \), find the Euler-Lagrange equations for \( \psi \) and \( \bar{\psi} \).

[4 marks]

(b) Considering \( \psi \) and \( \bar{\psi} \) as independent fields, find the momenta conjugate to \( \psi \) and \( \bar{\psi} \) for \( \mathcal{L} \) and \( \mathcal{L}' \). Show that the corresponding Hamiltonian densities are

\[
\mathcal{H} = \bar{\psi} \left( -i \vec{\gamma} \cdot \vec{\nabla} + m \right) \psi \quad \text{and} \quad \mathcal{H}' = -\frac{i}{2} \left( \bar{\psi} \vec{\gamma} \cdot \vec{\nabla} \psi - \left( \vec{\nabla} \bar{\psi} \right) \cdot \vec{\gamma} \psi \right) + m \bar{\psi} \psi.
\]

Next, explain why the Hamiltonians \( \mathcal{H} := \int d^3x \mathcal{H} \) and \( \mathcal{H}' := \int d^3x \mathcal{H}' \) derived from the Hamiltonian densities \( \mathcal{H} \) and \( \mathcal{H}' \) are in fact identical.

(Hint: compute the difference \( \mathcal{H} - \mathcal{H}' \) and show that \( \int d^3x (\mathcal{H} - \mathcal{H}') = 0 \) assuming that the fields vanish at infinity).

[6 marks]

(c) The charge of the Dirac field in terms of oscillators is given by the expression

\[
Q = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1}^{2} \left[ a_r^\dagger (p) a_r (p) + b_r (p) b_r^\dagger (p) \right].
\]

By introducing normal ordering, explain how one arrives at the expression for the corresponding normal-ordered quantity

\[
Q_{\text{norm ord}} = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1}^{2} \left[ a_r^\dagger (p) a_r (p) - b_r^\dagger (p) b_r (p) \right].
\]

Explain why one has to impose anticommutation relations among the oscillators unlike in the Klein-Gordon case. Use the expression for \( Q_{\text{norm ord}} \) to compute \( \langle 0 | Q_{\text{norm ord}} | 0 \rangle \).

[5 marks]

(d) Under a parity transformation, a Dirac spinor \( \psi(x) \) transforms as \( \psi(x) \rightarrow \gamma^0 \psi(x') \) where \( x'' := (t, -x) \). Show that under parity the Dirac bilinear \( A^\mu(x) := \bar{\psi}(x) \gamma^\mu \gamma^5 \psi(x) \) transforms as

\[
A^\mu(x) \rightarrow \begin{cases} 
-\bar{\psi}(x') \gamma^0 \gamma^5 \psi(x') & \text{if } \mu = 0, \\
\bar{\psi}(x') \gamma^i \gamma^5 \psi(x') & \text{if } \mu = i,
\end{cases}
\]

where \( i = 1, 2, 3 \) denotes the spatial index. You may use without proof the anticommutation relations \( \{ \gamma^\mu, \gamma^\nu \} = 2 \delta^{\mu\nu} \) and \( \{ \gamma^\mu, \gamma^5 \} = 0 \).

[5 marks]
Question 4: Noether’s theorem and dilatations

(a) Consider the theory of a massless real scalar \( \phi \), described by the Klein-Gordon Lagrangian

\[
L_0 = \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi).
\]

Determine the mass dimension of the field \( \phi \), and write down the equation of motion.  

[4 marks]

(b) Prove that the action \( S = \int d^4x \, L_0 \), where \( L_0 \) is given in part (a), is invariant under dilatations, which are defined as

\[
x' = e^{-a}x, \quad \phi'(x') = e^a \phi(x),
\]

where \( a \) is real. For this question you may find it more convenient to work at the level of the action and with finite transformations.  

[6 marks]

(c) For a theory with Lagrangian density \( L(\phi_r, \partial_\mu \phi_r) \) that is invariant under a continuous transformation, the conserved Noether current has the expression

\[
\delta J^\mu = \sum_r \frac{\partial L}{\partial (\partial_\mu \phi_r)} \delta \phi_r + L \delta x^\mu,
\]

where the sum over \( r \) is extended to all fields of the theory and, as usual \( \delta \phi_r = \phi_r'(x) - \phi_r(x) \) is the variation due to the transformation of the fields only. Using the formula for \( \delta J^\mu \) given above, find the Noether current associated to dilatations.  

(Hint: recall that \( \varphi(x) \rightarrow \varphi'(x') := \varphi(x) + \delta_T \varphi(x) \), where \( \delta_T \varphi(x) := \delta \varphi(x) + \delta x_\mu (\partial^\mu \varphi)(x) \).)

Next, check with an explicit calculation that the Noether current thus derived is conserved upon using the equations of motion.  

[6 marks]

(d) Finally, consider adding to the original Lagrangian the following two terms,

\[
L_m = -\frac{m^2}{2} \phi^2, \quad L_\lambda = -\lambda \phi^4,
\]

so that the resulting theory is now described by the action \( S = \int d^4x \, L_0 + \int d^4x \, L_m + \int d^4x \, L_\lambda \). Find for which values of the (constant) parameters \((m, \lambda)\) the theory is still invariant under dilatations. What are the dimensions of \( m \) and \( \lambda \)?  

[4 marks]
**Question 5: Interactions and the S-matrix**

(a) Consider the theory of a real scalar field with Lagrangian density $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$ where the free and interaction terms are given by

\[
\mathcal{L}_0 = \frac{1}{2} (\partial^\mu \phi) (\partial_\mu \phi) - \frac{m^2}{2} \phi^2, \quad \mathcal{L}_{\text{int}} = -\frac{\lambda}{3!} \phi^3.
\]

Write down the equation of motion for the field $\phi$ as derived from the total Lagrangian $\mathcal{L}$. What is the dimension of the coupling constant $\lambda$?

[4 marks]

(b) The Feynman propagator for the free $\phi$ field is given by

\[
i \Delta_F(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot x} \frac{i}{k^2 - m^2 + i\epsilon}.
\]

Prove the following two equations:

\[
(\Box + m^2) (i \Delta_F(x)) = -i \delta^{(4)}(x), \quad \text{and} \quad \Delta_F(-x) = \Delta_F(x).
\]

Explain the meaning of the $i\epsilon$ prescription in the above integral, thus rewriting $i \Delta_F(x)$ as a contour integral in the complex $k_0$ plane.

[6 marks]

(c) Write down the Dyson expansion for the $S$-matrix. For the case of $\mathcal{L}_{\text{int}}$ given in part (a), write down explicitly the first order term in $\lambda$.

[4 marks]

(d) Consider the initial state $|i\rangle := a^\dagger(k_1)|0\rangle$ and the final state $|f\rangle := a^\dagger(k_2)a^\dagger(k_3)|0\rangle$. Determine the matrix element $\langle f | S | i \rangle$ for this process to the first order in $\lambda$. In this question write the interaction term using normal ordering, i.e. use $\mathcal{L}_{\text{int}} = -\frac{\lambda}{3!} : \phi^3 :$ as the interaction term.

In this question you may use without proof the expansion for the fields in the interaction picture

\[
\phi(x) = \int \frac{d^3p}{(2\pi)^3(2E(p))} \left[ a(p)e^{-iE(p)t-\mathbf{p} \cdot \mathbf{x}} + b^\dagger(p)e^{iE(p)t-\mathbf{p} \cdot \mathbf{x}} \right],
\]

which is consistent with the commutation relations

\[
[a(p), a^\dagger(q)] = [b(p), b^\dagger(q)] = (2\pi)^3 2E(p) \delta^{(3)}(\mathbf{p} - \mathbf{q}),
\]

\[
[a(p), a(q)] = [b(p), b(q)] = [a(p), b(q)] = 0.
\]

[6 marks]
Formula Sheet: (in units where $\hbar = c = 1$)

- Four-vectors:
  
  $$a \cdot b = a_{\mu}b_{\mu} = a_{\mu}b^{\mu} = a_{\mu}b^{\nu}\eta_{\mu\nu} = a_{\mu}b_{\nu}\eta^{\mu\nu}$$

  with $\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

  $$x^{\mu} = (t, \vec{x}), \quad x_{\mu} = (t, -\vec{x}), \quad \partial^{\mu} = \frac{\partial}{\partial x^{\mu}} = \left( \frac{\partial}{\partial t}, -\vec{\nabla} \right), \quad \partial_{\mu} = \frac{\partial}{\partial x_{\mu}} = \left( \frac{\partial}{\partial t}, \vec{\nabla} \right), \quad \hat{p}^{\mu} = i\partial^{\mu}, \quad \hat{p}_{\mu} = i\partial_{\mu}$$

- Klein-Gordon equation:
  
  $$(-\hat{p} \cdot \hat{p} + m^2)\phi = (\partial_{\mu}\partial^{\mu} + m^2)\phi = 0$$

- Free Dirac equation in Hamiltonian form: $i\frac{\partial}{\partial t}\psi = (\vec{\alpha} \cdot \vec{p} + \beta m)\psi = (-i\vec{\alpha} \cdot \vec{\nabla} + \beta m)\psi$, or in covariant form:
  
  $$(i\partial - m)\psi = (i\gamma^{\mu}\partial_{\mu} - m)\psi = (\hat{p} - m)\psi = (\gamma \cdot \hat{p} - m)\psi = (\gamma^{\mu}\hat{p}_{\mu} - m)\psi = 0$$

- Dirac and Gamma matrices:
  
  $$(\alpha^i)^2 = 1, \quad \beta^2 = 1; \quad \alpha^i\alpha^j + \alpha^j\alpha^i = 0, \quad \beta\alpha^i = 0, \quad \alpha^i\beta + \beta\alpha^i = 0, \quad \gamma^0 = \beta, \quad \gamma^i = \beta\alpha^i, \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}1, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \text{with} \quad \{\gamma^5, \gamma^\mu\} = 0$$

- Dirac representation:
  
  $$\alpha^i = \sigma^1 \otimes \sigma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3, \quad \beta = \sigma^3 \otimes 1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^0 = \beta, \quad \gamma^i = \beta\alpha^i = i\sigma^2 \otimes \sigma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3, \quad \gamma_5 = \sigma^1 \otimes 1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5^2 = 1.$$  

  where the Pauli matrices are
  
  $$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

  Note that $\alpha^i, \beta$ and $\gamma^0$ are Hermitian, whereas the $\gamma^i$ are anti-Hermitian.