SPA 7010 : THE GALAXY

Solutions for Coursework 2

Solution 2.1 (Assessed Question)

Since the Plummer model has a potential $\Phi(r)$ that depends on radial distance r only, it is spherically symmetric.

The second Jeans equation for a spherically-symmetric potential is given as:

$$\frac{d}{dr}\left(n\langle v_r^2\rangle\right) + \frac{n}{r}\left[2\langle v_r^2\rangle - \langle v_\theta^2\rangle - \langle v_\phi^2\rangle\right] = -n\frac{d\Phi}{dr}$$

where the coords. are (r, θ, ϕ) and $n(r)$ is the number density of stars, v_r, v_θ, v_ϕ are the components of velocity in the r, θ and ϕ directions, respectively.

Given that there is no net rotation, and the velocity dispersion, σ, is constant and isotropic then:

$$\langle v_r^2\rangle = \langle v_\theta^2\rangle = \langle v_\phi^2\rangle = \sigma^2$$

and the Jeans equation becomes:

$$\frac{d}{dr}\left(n\sigma^2\right) + \frac{n}{r}\left[0\right] = -n\frac{d\Phi}{dr}$$

ie. $$\sigma^2\frac{dn}{dr} = -n\frac{d\Phi}{dr}$$

Integrating,

$$\sigma^2\int_{n_0}^{n(r)}\frac{dn}{n} = -\int_{\Phi(0)}^{\Phi(r)} d\Phi \qquad \text{where } n_0 = n(0)$$

$$\therefore \sigma^2\left[\ln n\right]_{n_0}^{n(r)} = -\Phi(r) + \Phi(0)$$

CW2 | Solution 2.1 (cont).

So, $$\sigma^2 \left(\ln n(r) - \ln n_0 \right) = \frac{GM_{tot}}{(r^2+a^2)^{1/2}} - \frac{GM_{tot}}{a}$$

where we have used $\Phi(r) = \frac{GM_{tot}}{(r^2+a^2)^{1/2}}$, for the Plummer potential

Simplifying further,

$$\sigma^2 \ln \left(\frac{n(r)}{n_0} \right) = \frac{GM_{tot}}{a} \left(\frac{1}{(1+r^2/a^2)^{1/2}} - 1 \right)$$

$$\therefore \quad \ln \left(\frac{n(r)}{n_0} \right) = \frac{GM_{tot}}{a\sigma^2} \left(\frac{1}{\sqrt{1+r^2/a^2}} - 1 \right)$$

and finally

$$n(r) = n_0 \exp \left[\frac{GM_{tot}}{a\sigma^2} \left(\frac{1}{\sqrt{1+r^2/a^2}} - 1 \right) \right]$$

An alternative approach, starting from $\sigma^2 \frac{dn}{dr} = -n \frac{d\Phi}{dr}$ gives:

$$\sigma^2 \frac{dn}{dr} = -n \frac{d\Phi}{dr} = -n \frac{GM_{tot} r}{(r^2+a^2)^{3/2}}$$

and integration gives

$$\sigma^2 \int_{n_0}^{n(r)} \frac{dn}{n} = -GM_{tot} \int_0^r \frac{r'}{(r'^2+a^2)^{3/2}} dr'$$

$$\therefore \quad \sigma^2 \left[\ln n \right]_{n_0}^{n(r)} = -GM_{tot} \left[-\frac{1}{(r'^2+a^2)^{1/2}} \right]_0^r$$

So, $$\sigma^2 (\ln n(r) - \ln n_0) = \frac{GM_{tot}}{(r^2+a^2)^{1/2}} - \frac{GM_{tot}}{a}$$

then proceed as above

Solution 2.2

(a) To calculate the mass $M(r)$ interior to a radius r we can use the equation

$$M(r) = \frac{r^2}{G}\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

which applies in cases of spherical symmetry.

Differentiating the expression for Φ in the question,

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{G\,M_{tot}\,r}{(r^2+a^2)^{3/2}}$$

which then gives:

$$M(r) = \frac{M_{tot}\,r^3}{(r^2+a^2)^{3/2}}$$

(We note that $\lim_{r\to\infty} M(r)$ does give M_{tot}, the total mass, as expected.)

(b) To determine the density ρ, we can consider a thin spherical shell of radius r and thickness $\mathrm{d}r$ centred on the mass distribution.

The volume of the shell is $4\pi r^2 \mathrm{d}r$ and its mass is $\mathrm{d}M = 4\pi r^2\rho(r)\mathrm{d}r$ where $\rho(r)$ is the density at a radius r

$$\therefore \quad \rho(r) = \frac{1}{4\pi\,r^2}\frac{\mathrm{d}M}{\mathrm{d}r}$$

(which is the equation of continuity of mass).

Differentiating the expression for $M(r)$, from part (a), using the product rule:

$$\frac{\mathrm{d}M}{\mathrm{d}r} = \frac{3\,M_{tot}\,r^2}{(r^2+a^2)^{3/2}} - \frac{3\,M_{tot}\,r^4}{(r^2+a^2)^{5/2}} = \frac{3\,M_{tot}\,a^2\,r^2}{(r^2+a^2)^{5/2}}$$

$$\therefore \quad \rho(r) = \frac{3\,M_{tot}}{4\pi}\,\frac{a^2}{(r^2+a^2)^{5/2}}$$

the result we had to prove.

[Alternative method for part (b):

Use Poisson's equation: $\nabla^2\Phi = 4\pi G\rho$

Poisson's equation in this case of spherical symmetry gives:

$$\rho(r) = \frac{1}{4\pi G}\,\nabla^2\Phi = \frac{1}{4\pi G}\,\frac{1}{r^2}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)$$

Substituting for the expression for $\mathrm{d}\Phi/\mathrm{d}r$ from part(a) i.e.

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{G\,M_{tot}\,r}{(r^2+a^2)^{3/2}}$$

Gives:

$$\rho(r) = \frac{1}{4\pi G r^2}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{G\,M_{tot}\,r}{(r^2+a^2)^{3/2}}\right) = \frac{1}{4\pi r^2}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{M_{tot}\,r^3}{(r^2+a^2)^{3/2}}\right)$$

$$\therefore \quad \rho(r) = \frac{1}{4\pi r^2}\left[\frac{3\,M_{tot}\,r^2}{(r^2+a^2)^{3/2}} - \frac{3\,M_{tot}\,r^4}{(r^2+a^2)^{5/2}}\right] = \frac{3\,M_{tot}}{4\pi}\,\frac{a^2}{(r^2+a^2)^{5/2}}$$

(c) The constant a can be used to flatten the potential in the core region, since when $a \ll r$, $\Phi(r) \rightarrow$ constant.