SPA7010U/SPA7010P: THE GALAXY

Solutions for Coursework 1

Questions distributed on: 6 February 2020.

Solution 1.1 [**Assessed question**]

(a) We're told that this is a faint galaxy, so essentially we have to try to classify it based on its spectral characteristics alone. The galaxy has a strong continuum with absorption lines and some emission lines superimposed. The emission lines are a characteristic of areas of star formation. This really rules out an elliptical galaxy, since ellipticals typically have no active star formation. The combination of a strong continuum with both absorption and emission lines suggests that this is most likely to be a spiral galaxy.

(b) Very strong emission lines on a stellar continuum is a typical characteristic of an irregular galaxy. So based on the spectral characteristics alone, this is probably an irregular galaxy. It could be another type of galaxy, such as a spiral, experiencing a strong burst of star formation because of an interaction with another galaxy, but we are not considering merger processes in this course, and in the absence of other information, the simplest explanation is that is an irregular galaxy.

The very strong emission lines indicate strong star formation. The star formation rate is stronger in this galaxy compared to the galaxy in part (a).

(c) When two bodies interact, the interaction is collisional if the interactions between the individual particles in the bodies affect the motions substantially. The interactions are collisionless if the interactions between the individual particles do not affect the motions.

The interactions between two gas clouds are collisional.

(d) The merging of the two small systems of stars is collisional because the interactions between individual stars are important in this case. This is different from the case of two merging galaxies: the star-star encounters are not important in the merger of two galaxies, so this would be a collisionless process.

(e) Some of the total mechanical energy (potential + kinetic) is converted into heat, so the mechanical energy is not conserved and the collapse is dissipative.

[**continued ...**]

(f) Integrating over the surface brightness gives the luminosity of a galaxy to be $L \propto I_0 R_0^2$

Because I_0 is constant for all galaxies of this type, $L \propto R_0^2$ for all

The virial theorem implies $M/R_0 \propto v^2$, where M is the mass of a galaxy and v is a typical velocity of stars in a galaxy

Eliminating R_0 gives $L \propto M^2 v^{-4}$

Since the mass-to-light ratio $M/L =$ constant, $M \propto L$

Substituting for M in $L \propto M^2 v^{-4}$ gives $L \propto L^2 v^{-4}$ so finally: $L \propto v^4$

This is the same as the Tully-Fisher relation for spiral galaxies, or the Faber-Jackson relation for elliptical galaxies.

End of assessed question

Solution 1.2

(a) To find $M(r)$, consider a thin spherical shell of radius r and thickness dr concentric with the galaxy. The mass in the shell will be

$$\mathrm{d}M = 4\pi\, r^{\,2}\, \mathrm{d}r\; \rho(r)$$

(this is the equation of continuity of mass).

Integrating from the centre of the galaxy (radius = 0) to a radial distance r,

$$\int_0^{M(r)} \mathrm{d}M' = \int_0^r 4\pi\, r'^{\,2}\, \mathrm{d}r'\; \rho(r') = \int_0^r 4\pi\, r'^{\,2}\, \mathrm{d}r'\; \frac{q\, a}{4\pi}\, \frac{r'^q}{r'^3\,(r'+a)^{q+1}}\; M_{tot}\ .$$

$$\therefore\ M(r) = q\, a\, M_{tot} \int_0^r \frac{r'^{q-1}}{(r'+a)^{q+1}}\, \mathrm{d}r' = M_{tot}\left[\frac{r'^q}{(r'+a)^q}\right]_{r'=0}^{r}$$

and using the standard integral provided.

This gives,

$$M(r) = M_{tot}\left(\frac{r^q}{(r+a)^q} - \frac{0^q}{(0+a)^q}\right) = M_{tot}\, \frac{r^q}{(r+a)^q}\ ,$$

the required result (for all $q \neq 0$).

(b) We need to calculate the potential $\Phi(r)$ for $q = 1$ (the Jaffe model). The simplest way to do this is to use

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{GM(r)}{r^2}\ .$$

From the answer to the first part of the question, putting $q = 1$, the mass $M(r)$ interior to a radius r is

$$M(r) = M_{tot}\, \frac{r}{(r+a)} \qquad \text{for } q = 1.$$

Therefore,

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{G}{r^2}\, M_{tot}\, \frac{r}{(r+a)} = \frac{GM_{tot}}{r\,(r+a)}\ .$$

Integrating from a radius r to infinity (remembering that $\Phi(\infty) = 0$ from the definition of gravitational potential),

$$\int_{\Phi(r)}^{0} \mathrm{d}\Phi' = GM_{tot} \int_r^{\infty} \frac{1}{r'\,(r'+a)}\, \mathrm{d}r'\ .$$

This can be solved using partial fractions:

$$\begin{aligned}
0 \; - \; \Phi(r) &= G\,M_{tot} \int_r^\infty \frac{1}{a} \left(\frac{1}{r'} \; - \; \frac{1}{(r'+a)} \right) \mathrm{d}r' \\
-\;\Phi(r) &= \frac{G\,M_{tot}}{a} \left[\ln r' \; - \; \ln(r'+a) \; \right]_{r'=r}^{\infty} \\
&= \frac{G\,M_{tot}}{a} \left[\ln\left(\frac{1}{1+a/r'} \right) \right]_r^\infty \;\; = \;\; - \, \frac{G\,M_{tot}}{a} \ln\left(\frac{r}{r+a} \right) \\
&= \frac{G\,M_{tot}}{a} \ln\left(\frac{r+a}{r} \right) \quad ,
\end{aligned}$$

which gives

$$\Phi(r) \;=\; - \, \frac{G\,M_{tot}}{a} \ln\left(\frac{r+a}{r} \right)$$

for the potential at a radius r in the Jaffe ($q = 1$) model.

(c) If $q \to 0$, the density profile gives $\rho = 0$ for $r > 0$. However,

$$\rho(0) \;=\; \frac{a\,M_{tot}}{4\pi} \lim_{q,\,r \to 0} \frac{q\,r^q}{r^3\,(r+a)^{q+1}} \quad .$$

So $q \to 0$ implies that all the mass M_{tot} is concentrated at the centre: it corresponds to a point mass.

Solution 1.3

(a) The density depends on radius r and does not depend on angle, so the profile is spherically symmetric.

Therefore the equation of continuity of mass $\mathrm{d}M/\mathrm{d}r = 4\pi r^2 \rho$ applies.

Substituting for the profile ρ,

$$\frac{\mathrm{d}M}{\mathrm{d}r} \;=\; 4\pi r^2 \, \frac{k}{r(r+a)^2} \quad .$$

Integrating from the centre to radius r,

$$\int_0^{M(r)} \mathrm{d}M \;=\; \int_0^r 4\pi r'^2 \, \frac{k}{r'(r'+a)^2} \, \mathrm{d}r' \qquad \therefore \;\; M(r) \;=\; 4\pi k \int_0^r \frac{r'}{(r'+a)^2} \, \mathrm{d}r' \quad .$$

This integral can be solved using

$$\int \frac{r}{(r+a)^2}\,\mathrm{d}r \;=\; \int \frac{r+a-a}{(r+a)^2}\,\mathrm{d}r \;=\; \int \frac{r+a}{(r+a)^2}\,\mathrm{d}r \;-\; a\int \frac{1}{(r+a)^2}\,\mathrm{d}r$$

$$= \int \frac{1}{(r+a)}\,\mathrm{d}r \;+\; a\,\frac{1}{(r+a)} \;=\; \ln(r+a) \;+\; \frac{a}{(r+a)} \;+\; c\;.$$

$$\therefore\;\; M(r) \;=\; 4\pi k\left[\ln(r'+a) \;+\; \frac{a}{(r'+a)}\right]_0^r$$

$$= 4\pi k\left(\ln(r+a) \;+\; \frac{a}{(r+a)} \;-\; \ln(a) \;-\; \frac{a}{(a)}\right)$$

$$\text{So}\qquad M(r) \;=\; 4\pi k\left(\ln\left(\frac{r}{a}+1\right) \;-\; \frac{r}{(r+a)}\right)\;,$$

which is the mass inside a radius r that the question asks for.

(b) As $r \to \infty$, $M(r) \to \infty$. So the model is not physically realistic at large radii.

(c) To calculate the potential, we make use of the spherical symmetry once more.

For spherical symmetry, $\dfrac{GM(r)}{r^2} \;=\; \dfrac{\mathrm{d}\Phi}{\mathrm{d}r}$.

Substituting for $M(r)$ and integrating from infinity to radius r,

$$\int_0^{\Phi(r)} \mathrm{d}\Phi \;=\; \int_\infty^r \frac{4\pi Gk}{r^2}\left(\ln\left(\frac{r}{a}+1\right) \;-\; \frac{r}{(r+a)}\right)\mathrm{d}r$$

because the potential is 0 at $r \to \infty$.

Therefore,

$$\Phi(r) - 0 \;=\; 4\pi Gk \int_\infty^r \left(\frac{1}{r^2}\,\ln\left(\frac{r}{a}+1\right) \;-\; \frac{1}{r(r+a)}\right)\mathrm{d}r$$

The integrals can be solved using partial fractions:

$$\int \frac{1}{r(r+a)}\,\mathrm{d}r \;=\; \frac{1}{a}\int\left(\frac{1}{r} - \frac{1}{r+a}\right)\mathrm{d}r$$

$$= \frac{1}{a}\Big(\ln r - \ln(r+a)\Big) \;+\; c \;=\; \frac{1}{a}\ln\left(\frac{r}{r+a}\right) \;+\; c$$

Then, using integration by parts,

$$\int \frac{1}{r^2} \ln\left(\frac{r+a}{a}\right) \mathrm{d}r \;=\; -\frac{1}{r}\ln\left(\frac{r+a}{a}\right) \;+\; \int \frac{1}{r}\,\frac{1}{r+a}\,\mathrm{d}r$$

$$= -\frac{1}{r}\ln\left(\frac{r+a}{a}\right) + \frac{1}{a}\ln\left(\frac{r}{r+a}\right) + c \qquad \text{using the integral above.}$$

Using these integrals, we get for the potential

$$\begin{aligned}\Phi(r) &= 4\pi Gk\left[-\frac{1}{r}\ln\left(\frac{r+a}{a}\right) + \frac{1}{a}\ln\left(\frac{r}{r+a}\right) - \frac{1}{a}\ln\left(\frac{r}{r+a}\right)\right]_{\infty}^{r} \\ &= 4\pi Gk\left[-\frac{1}{r}\ln\left(\frac{r+a}{a}\right)\right]_{\infty}^{r} = 4\pi Gk\left(-\frac{1}{r}\ln\left(\frac{r+a}{a}\right) - 0\right)\end{aligned}$$

So the potential at a distance r from the centre is

$$\Phi(r) \;=\; -\frac{4\pi Gk}{r}\ln\left(\frac{r}{a}+1\right) \; .$$

(d) The central density is infinite. This appears not to be physically realistic.

(However, one of the people who first used this profile to describe galaxies – Carlos Frenk – has argued that the profile might be realistic in the cores of galaxies after all. He was thinking about the possibility that most massive galaxies have black holes – and therefore a singularity – at their cores.)