EBU5375 Signals and Systems: 
Analysis and synthesis equations in Matlab

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Background

Given a periodic discrete-time signal $x_N[n]$ of period $N$:

1. Its **fundamental frequency** is $\Omega_0 = \frac{2\pi}{N}$.

2. According to the **synthesis equation**, $x_N[n]$ can be expressed as the sum of $N$ harmonically related complex exponentials of frequencies $\Omega_k = k\frac{2\pi}{N}$:

$$x_N[n] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n}$$

3. The Fourier coefficients $a_k$ can be determined by using the **analysis equation** as

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x_N[n] e^{-jk\Omega_0 n}$$
In order to obtain the Fourier series decomposition of a periodic signal $x_N[n]$ of period $N$, we need to:

1. Identify the **fundamental frequency** $\Omega_0$.
2. Determine the $N$ **harmonic frequencies** $\Omega_k = k\Omega_0$.
3. Obtain the **Fourier coefficients** $a_k$. 

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Determining the fundamental frequency $\Omega_0$ and its harmonics $\Omega_k$ is very easy.

The Fourier coefficients $a_k$ can be obtained analytically. For instance, for the periodic square wave of period $N$ defined within one period centred around $n = 0$ as:

$$x_N[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & \text{otherwise} \end{cases}$$

we can obtain its Fourier coefficients are:

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-j k \Omega_0 n} = \begin{cases} \frac{2N_1+1}{N}, & k = 0 \\ \frac{1}{N} \frac{\sin[2\pi k (N_1+1/2)/N]}{\sin(\pi k/N)}, & k \neq 0 \end{cases}$$
You will have noticed that the analysis equation in DT:

\[ a_k = \frac{1}{N} \sum_{n=0}^{N-1} x_N[n] e^{-j k \Omega_0 n} \]

is essentially a mathematical operation in which we:

1. Calculate \( N \) products of complex numbers.
2. Add \( N \) complex numbers.

Computers are very good at doing additions and multiplications so, why don’t we let computers calculate the Fourier coefficients for us?
Objectives of the lab

In this lab, we will obtain numerically the Fourier coefficients of a periodic discrete-time signal and then we will synthesise the signal by using its complex exponential components.

We will:

1. Define a square wave and identify its fundamental frequency.
2. Obtain numerically the Fourier coefficients $a_k$.
3. Plot the coefficients $a_k$.
4. Synthesise the periodic square wave as a Fourier series.
5. Lowpass filter the signal.
In this lab, we will work with the periodic square wave $x[n]$ with period $N = 21$ defined as:

$$
x[n] = \begin{cases} 
1, & |n| \leq N_1 \\
0, & N_1 < |n| \leq 10 
\end{cases}
$$

Your job now is to

- Identify its fundamental frequency $\Omega_0$ and its harmonics $\Omega_k$.
- Draw on a piece of paper three periods of $x[n]$. We will assume $N_1 = 2$.
Step 2: Obtaining the Fourier coefficients

The following lines of code calculate the Fourier coefficients $a_k$ of $x[n]$:

```matlab
n_s = -10;
n_e = 10;
n = n_s:1:n_e; % Definition of the time vector
x = zeros(size(n));
x(9:13)=1; % Definition of x[n]
N = 21; % Period of x[n]
omega_0=2*pi/N; % Fundamental frequency of x[n]

a_k=zeros(1,N);
for k=0:20 % This loop calculates the Fourier coefficients a_k
  a_k(k+1)=(1/N)*sum(x.*exp(-k*1i*omega_0*n));
end
```

Your job now is to

- Plot the signal $x[n]$ in the time interval $-10 \leq n \leq 10$.
- Identify the analysis equation in the previous lines of code.

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Step 3: Plotting the Fourier coefficients

The following lines of code plot the magnitude of the coefficients $a_k$.

```matlab
figure
stem([0:10,-10:-1],abs(a_k)) % plots $a_k$ against $n$
xlabel('k') % adds text below the X-axis
ylabel('a_k') % adds text beside the Y-axis
```

Your job now is to

- Identify and justify the shape of the magnitude of the coefficients $a_k$.
- Understand how we plot $a_k$ against $k$ by using `stem`. 
Step 4: Synthesising $x[n]$

The following lines of code synthesise 3 periods of $x[n]$ by using the Synthesis equation:

```matlab
ns = -31;
ne = 31;
n = ns:1:ne; % New time vector
xsyn = zeros(size(n));
k=0:20;
for m=-31:31 % Synthesis of x
    xsyn(m+32)=sum(a_k.*exp(k*1i*omega_0*m));
end
```

Your job now is to
- Plot the signal $x[n]$ in the time interval $-31 \leq n \leq 31$.
- Identify the synthesis equation in the previous lines of code.
Step 4: Lowpass filtering $x[n]$

In the following lines of code, we filter out the frequencies $|\Omega| > 2\pi/21$ of $x[n]$, producing the signal $y[n]$ with Fourier coefficients $b_k$:

```matlab
b_k=zeros(size(a_k));
b_k(1)=a_k(1);
b_k(2)=a_k(2);
b_k(21)=a_k(21);
for m=-31:31 % Synthesis of x
    y(m+32)=sum(b_k.*exp(k*1i*omega_0*m));
end

figure
stem(n,y) % plots a_k against n
xlabel('n') % adds text below the X-axis
ylabel('y') % adds text beside the Y-axis
axis tight
```

Your job now is to

- Plot the signal $y[n]$ in the time interval $-31 \leq n \leq 31$.
- How have we implemented the filter $x[n]$?
Step 5: Changes in the duty cycle

The duty cycle of the proposed discrete-time periodic signal is
\[ \rho = \frac{2N_1 + 1}{21}. \]
For instance, for \( N_1 = 2 \), the duty cycle is \( \rho = 5/21 \).

Your job now is to

- Plot the \( x[n] \) for \( \rho = 1/21, 3/21, 11/21, 21/21 \).
- Plot the Fourier coefficients \( a_k \) for \( \rho = 1/21, 3/21, 11/21, 21/21 \).
Discuss your results.