Solution of the diffusion equation in 1D

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} \qquad 0 \leq x \leq \ell \tag{1}$$

1 Steady state

Setting $\partial C/\partial t = 0$ we obtain

$$\frac{d^2 C}{dx^2} = 0 \qquad \Rightarrow C_s = ax + b$$

We determine a, b from the boundary conditions.

$$C\left(0\right) = C_1, \qquad C\left(\ell\right) = C_2 \tag{2}$$

It follows that

$$b = C_1, \qquad a = \frac{C_2 - C_1}{\ell}$$

$$\begin{aligned} C_s\left(x\right) &= \frac{C_2 - C_1}{\ell}x + C_1 \\ \text{Flux} &= -D\frac{\partial C_s}{\partial x} = \frac{C_1 - C_2}{\ell} \end{aligned} \tag{3}$$

2 Time-dependent solutions

We choose again the boundary conditions (2) and

$$C(x,0) = C_0(x) \tag{4}$$

as initial condition. It is convenient to consider the excess quantity

$$u(x,t) = C(x,t) - C_s(x) \tag{5a}$$

Using (1)-(3) we see that u satisfies

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \tag{5b}$$

with

$$\begin{aligned} u(0) &= u(\ell) = 0 && (5c) \\ u(x,0) &= C_0 - C_s(x) \\ &\equiv u_0(x) && (5d) \end{aligned}$$

Let ϕ_m be the eigenfunctions of the diffusion operator d^2/dx^2. Since the operator is dissipative, the correponding eigenvalues are non-positive. We denote them by $-k_m^2$ (k real)

$$\frac{d^2\phi_m(x)}{dx^2} = -k_m^2\phi_m(x) \tag{6}$$

Any function of the form $u = A_m(t)\,\phi_m$ satisfies then eq.(5), provided that $A_m(t)$ satisfies the ordinary differential equation

$$\frac{dA_m}{dt} = -Dk_m^2 A_m \tag{7a}$$

or

$$A_m(t) = A_m(0)\,\mathrm{e}^{-Dk_m^2 t} \tag{7b}$$

On the other hand, in general, functions u of this form do not satisfy the initial condition. To satisfy this condition we seek for solutions in the form of an infinite series of ϕ_m's (this is legitimate since the equation is linear)

$$u\left(x,t\right)=\sum_{m}A_{m}\left(t\right)\phi_{m}\left(x\right) \tag{8}$$

and fix the $A_{m}\left(0\right)$'s by requiring that

$$\sum_{m}A_{m}\left(0\right)\phi_{m}\left(x\right)=u\left(x,0\right)=u_{0}\left(x\right) \tag{9}$$

To compute $A_{m}\left(0\right)$ we use the orthogonality property of ϕ_{m}, guaranted by the fact that the diffusion operator is self-adjoint :

$$\begin{aligned}\int_{0}^{\ell}dx\phi_{n}^{*}(x)\phi_{m}(x) &= 0 \quad n\neq m \\ &= N_{n} \quad n=m\end{aligned} \tag{10}$$

Multiplying both sides of (9) by $\phi_{m}^{*}(x)$ and integrating over x we thus obtain

$$A_{n}(0)=\frac{\int_{0}^{\ell}dx\phi_{n}^{*}(x)u_{0}(x)}{N_{n}} \tag{11}$$

which combined with (7b) and (8) yields the solution

$$u(x,t)=\sum_{m}\frac{\int_{0}^{\ell}dx\phi_{n}^{*}(x)u_{0}(x)}{N_{m}}\phi_{m}(x) \tag{12}$$

We now compute ϕ_{m} and k_{m} explicitly for the boundary conditions (5b). On inspecting (6) we see that ϕ_{m} must be of the form

$$\phi_{m}(x)=C\cos k_{m}x+D\sin k_{m}x$$

Applied to $x=0$ and $x=\ell$ this leads to

$$\begin{aligned}0 &= \phi_{m}(0)=C \\ 0 &= \phi_{m}(\ell)=D\sin k_{m}\ell\end{aligned}$$

implying that $k_m \ell = m\pi$ (m integer) or

$$\begin{aligned} k_m &= \frac{m\pi}{\ell} \\ \phi_m &= \sin\frac{m\pi x}{\ell} \qquad \text{(up to a factor)} \end{aligned} \tag{13}$$

the full solution $u(x,t)$ being (cf. eq. (12)), with $N_m = \ell/2$

$$u(x,t) = \sum_m \frac{2}{\ell}\left(\int_o^\ell dx \sin\frac{m\pi x}{\ell} u_0(x)\right) \sin\frac{m\pi x}{\ell} \mathrm{e}^{-D\frac{m^2\pi^2}{\ell^2}t} \tag{14}$$

(Fourrier series)

To complete the evaluation suppose that $u_0(x) = u_0 = \text{constant}$, then,

$$\begin{aligned} u_0 \int_0^\ell dx \sin\frac{m\pi x}{\ell} &= 0 \qquad \text{if } m \text{ is even} \\ &= \frac{2u_o\ell}{\pi m} \qquad \text{if } m \text{ is odd} = 2n+1 \end{aligned} \tag{15}$$

Finally,

$$u(x,t) = \frac{4u_0}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\mathrm{e}^{-D\frac{(2n+1)^2 ?pi^2}{\ell^2}t}\sin\frac{(2n+1)\pi x}{\ell} \tag{16}$$

Notice that $u(x,t) \to 0$ as $t \to \infty$ with a characteristic time $\bar{t} = \frac{\ell^2}{D\pi^2}$