To check the Claim, note that the RHS of $(*)$ is

$$h \circ \sigma (b_1 b_2 b_3 \dots)$$

$$= h(b_2 b_3 b_4 \dots)$$

$$= \frac{b_2}{2} + \frac{b_3}{4} + \frac{b_4}{8} + \dots$$

LHS of $(*)$ is

$$D \circ h (b_1 b_2 b_3 \dots)$$

$$\cancel{= D(b_2 b_3 b_4 \dots)}$$

$$= D\left(\frac{b_1}{2} + \frac{b_2}{4} + \frac{b_3}{8} + \dots \right)$$

$$= b_1 + \frac{b_2}{2} + \frac{b_3}{4} + \frac{b_4}{8} + \dots \pmod 1$$

$$= \frac{b_2}{2} + \frac{b_3}{4} + \frac{b_4}{8} + \dots = \text{RHS}.$$

Note that h is not a bijection,
since it is not injective
(i.e. it is not one-to-one),
as we see with the following example:

$$h(0\bar{1}) = h(0111111\ldots)$$

$$= \frac{0}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

$$= \frac{1}{4} \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i$$

$$= \frac{1}{4} \frac{1}{1 - \frac{1}{2}}$$

$$= \frac{1}{4} 2$$

$$= \frac{1}{2}$$

Note that h is not a bijection, since it is not injective (i.e. it is not one-to-one) as we see with the following example:

$$h(0\bar{1}) = h(0111111111\ldots)$$

$$= 0/2 + 1/4 + 1/8 + 1/16 + \ldots$$

$$= 1/4 \left(\sum_{i=0}^{\infty} (1/2)^i \right)$$

$$= 1/4 \frac{1}{1 - 1/2}$$

$$= 1/4 \times 2$$

$$= 1/2,$$

but also

$$h(1\overline{0}) = h(1\,00000\ldots)$$

$$= \tfrac{1}{2} + \tfrac{0}{4} + \tfrac{0}{8} + \tfrac{0}{16} + \tfrac{0}{32} + \ldots$$

$$= \tfrac{1}{2}$$

$$= h(0\overline{1})$$

So h is not injective.

(compare this to the familiar fact that in decimal notation, the number 1 has the two representations $1 = 1\cdot 0000\ldots$
$= 0\cdot 9999\ldots$)

Chaos (What might it mean?)

The doubling map has the property that its periodic points are <u>dense</u> in $[0,1)$

i.e. For any $x \in [0,1)$ and any $\varepsilon > 0$ there exists a periodic point y with $|x-y| < \varepsilon$

(Recall that rational numbers of the form $\frac{p}{2^n-1}$ are periodic (of period n) under D).

The doubling map D also has the property that there is a <u>dense orbit</u>,

i.e. there is a point $x_0 \in [0,1)$ such that

$$\mathcal{O}(x_0) = \{ D^i(x_0) \; : \; i = 0,1,2,3,\ldots \}$$

is dense in $[0,1)$.

i.e. For any $x \in [0,1)$, and for any $\varepsilon > 0$ there exists $i \geq 0$ such that $| D^i(x_0) - x | < \varepsilon$

To justify the existence of such a dense orbit, we work symbolically, and let

$$S = \underbrace{\underline{0}\ \underline{1}}_{\text{length-1 blocks}}\ \underbrace{\underline{00}\ \underline{01}\ \underline{10}\ \underline{11}}_{\text{length-2 blocks}}\ \underbrace{\underline{000}\ \underline{001}\ \underline{010}\ \underline{011}\ \underline{100}\ \underline{101}\ \underline{110}\ \underline{111}}_{\text{length-3 blocks}}$$

$$\underline{0000}\ \underline{0001}\ \underline{0010}\ \underline{0011}\ \underline{0100}\ \underline{0101}\ \underline{0110}\ \underline{0111}\ \underline{1000}\ \underline{1001}\ \underline{1010}\ \underline{1011}\ \underline{1100}\ \underline{1101}\ \underline{1110}\ \underline{1111}$$

...

where we build up the sequence S by first writing down all length-1 'blocks' then — — — — — — 2 — —,
— — — — — — 3 — —
— — — — — — 4 — —
etc
⋮
— — — — — — n — —.

Let $x_0 = h(s)$

$$= \frac{0}{2} + \frac{1}{4} + \frac{0}{2^3} + \frac{0}{2^4} + \frac{0}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \dots$$

We claim that x_0 has a dense orbit under the doubling map D!

To justify this claim, choose some (other) $x \in [0,1)$, and suppose x has binary expansion

$$b(x) = b_1 b_2 b_3 b_4 \dots$$

(i.e. $h(b_1 b_2 b_3 b_4 \dots) = x$)

Then it is possible ~~to~~ (due to the recipe for defining s) to choose some $i \in \mathbb{N}$ such that $\sigma^i(s)$ and $b(x)$ begin with a common (long) block of digits/symbols i.e. $\sigma^i(s) = \underbrace{b_1 b_2 \dots b_k}_{\text{same symbols as } b(x) \text{ begins with}} c_{k+1} c_{k+2} \dots$

This ensures that

$$| D^i(x_0) - x | < \frac{1}{2^k} < \varepsilon$$

if k is chosen sufficiently large

So x_0 does have a dense orbit under D.

The tent map T, and the logistic map f_4 both also have the property that ~~exa~~ there exist points with dense orbits.

Although there is no agreed definition of 'chaos', there is some universal agreement that an indication of chaotic dynamics is <u>sensitive dependence on initial conditions</u> (SDIC):

<u>Defn</u> Let $I \subseteq \mathbb{R}$ be an interval, and consider $f: I \to I$. We say that f has <u>sensitive dependence on initial conditions</u> (SDIC) at $x \in I$ if there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists $y \in I$ with $|x - y| < \delta$ and there exists $n \in \mathbb{N}$ such that $|f^n(x) - f^n(y)| > \varepsilon$.

We say that $f: I \to I$ has SDIC if it has SDIC at x, for every $x \in I$.

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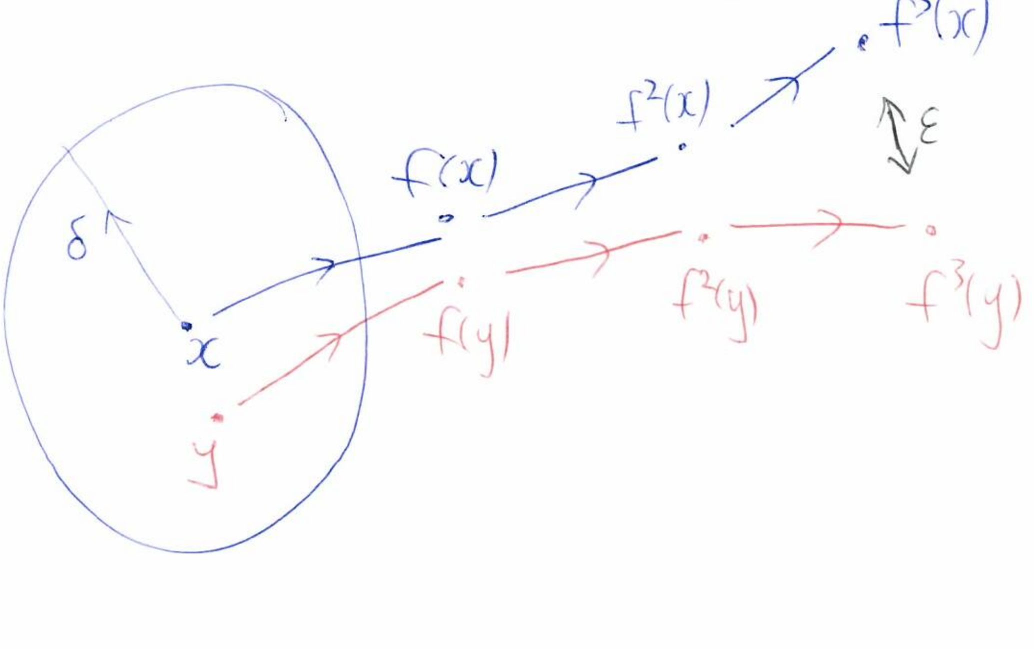
i.e. There are points arbitrarily close to x whose orbits diverge / separate from the orbit of x (by at least ε).



<u>Proposition</u> The doubling map $D: [0,1) \circlearrowleft$ (given by $D(x) = 2x \pmod 1$) has SDIC at every $x \in [0,1)$ (ie. The map D has SDIC).

<u>Proof</u> Let $\varepsilon = 1/3$. Suppose $x \in [0,1)$. Given $\delta > 0$, choose $n \in \mathbb{N}$ such that $1/2^{n+1} < \delta$, and let

$$y = \begin{cases} x + \frac{1}{2^{n+1}} & \text{if } D^n(x) \in [0, 1/2) \\ x - \frac{1}{2^{n+1}} & \text{if } D^n(x) \in [1/2, 1) \end{cases}$$

Then $|y - x| = \frac{1}{2^{n+1}} < \delta$

and

$$|D^n(x) - D^n(y)| = 1/2 > 1/3 = \varepsilon.$$
$\square$

<u>Proposition</u> The tent map $T : [0,1] \circlearrowleft$
(given by $T(x) = \begin{cases} 2x & \text{if } 0 \leq x < 1/2 \\ 2-2x & \text{if } 1/2 \leq x \leq 1 \end{cases}$)
has SDIC.

<u>Lemma</u> If $f : I \to I$ has SDIC, and is topologically conjugate to some $g : I \to I$, then g has SDIC.

Consequently:

<u>Prop</u> The logistic map $f_4 : [0,1] \circlearrowleft$ (defined by $f_4(x) = 4x(1-x)$) has SDIC.

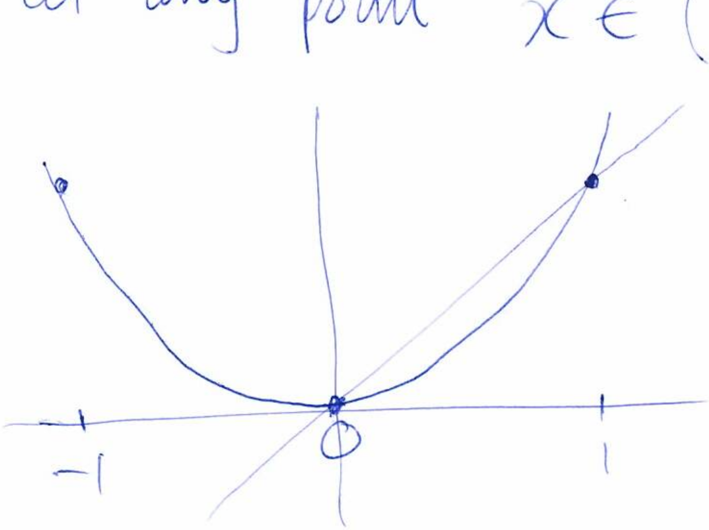
<u>Remark</u> (a) A map $f:\mathbb{R}\to\mathbb{R}$ of the form $f(x) = x + c$ $(c\in\mathbb{R})$ does <u>not</u> have SDIC at <u>any</u> point

(if $|x-y| = \delta$ then

$$|f^n(x) - f^n(y)| = \delta$$

for all $n \geq 0$)

(b) The map $f(x)=x^2$ does <u>not</u> have SDIC at any point $x\in(-1,1)$



Definition (from Robert Devaney's book "An introduction to chaotic dynamical systems")

Let $I \subseteq \mathbb{R}$ be an interval.

We say that $f : I \to I$ is chaotic (in the sense of Devaney) if

(i) f has SDIC (at all points in I)

(ii) the set of all periodic points of f is a dense subset of I.

(iii) there is an orbit (under f) of some point which is a dense subset of I

This definition of Devaney dates back to the mid-1980s.

It was later shown (in the mid-1990s) that if $f: I \to I$ is <u>continuous</u> then property (iii) implies (i) and (ii).

So the definition of 'chaotic' (in the sense of Devaney) can be reduced to saying that some point has a dense orbit under f (<u>if</u> f is continuous).

<u>Examples</u>: The following maps are chaotic in the sense of Devaney:

- The doubling map $D: [0,1) \to [0,1)$
- The tent map $T: [0,1] \to [0,1]$
- The logistic map $f_4: [0,1] \to [0,1]$
  $: x \mapsto 4x(1-x)$.

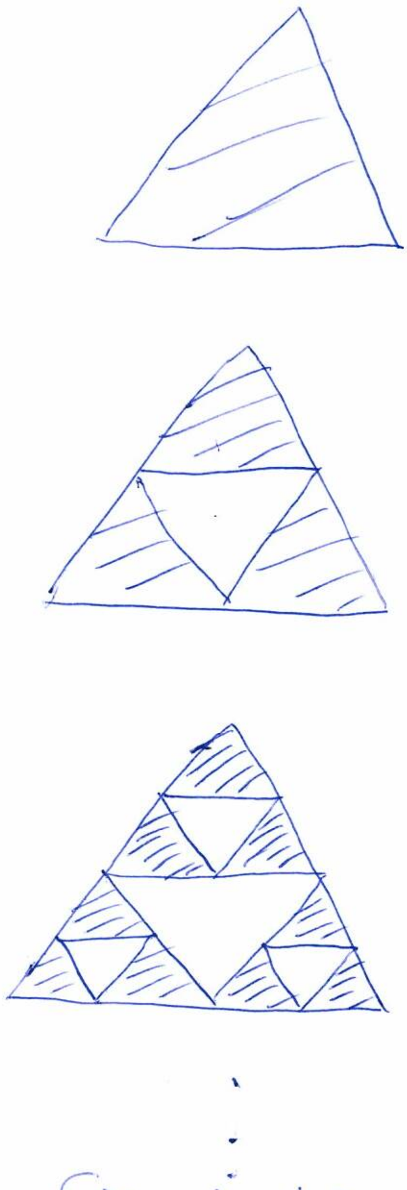
Fractals

Pictorial motivation

Step 0

Step 1

Step 2



"Middle-Third Cantor set"

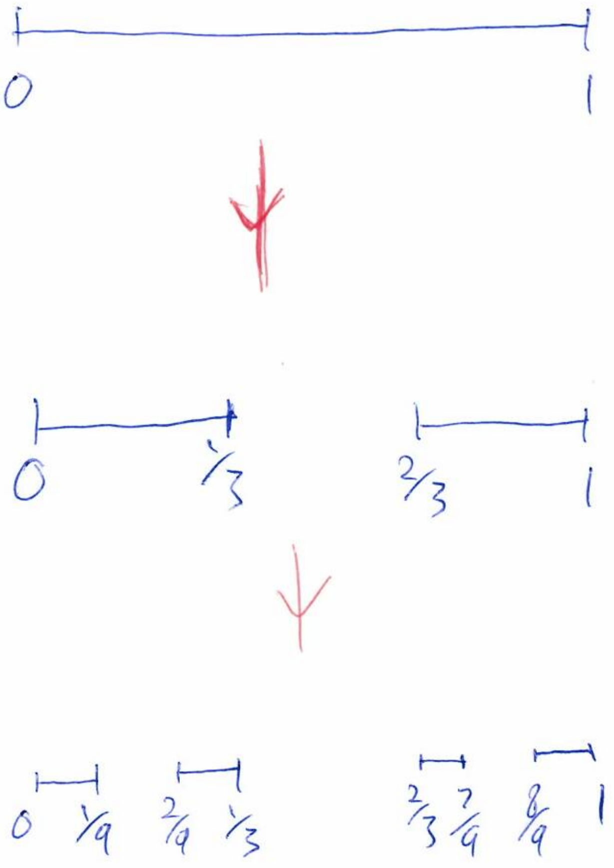
Sierpinski Triangle

These objects are obtained "in the limit" as the number of steps tends to ∞

(Middle-third) Cantor set

We shall now construct, in more detail, the <u>middle-third Cantor set</u>.

Let $C_0 = [0,1] \subseteq \mathbb{R}$, and consider the following inductive definition of C_n:



$C_0 = [0,1]$

(Remove the open 'middle third' $(1/3, 2/3)$ of this interval)

$C_1 = [0, 1/3] \cup [2/3, 1]$

Remove the open middle third of the 2 disjoint intervals making up C_1

$$C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$$

We continue this indefinitely, and at step n:

$$C_n = C_{n-1} \setminus \left\{ \begin{array}{c} \text{open middle-thirds} \\ \text{of all sub-intervals} \\ \text{of } C_{n-1} \end{array} \right\}$$

Note that C_n is a union of 2^n disjoint closed intervals of the form

$$\left[\frac{k}{3^n}, \frac{k+1}{3^n} \right] \quad \left(\begin{array}{c} \text{these intervals all} \\ \text{have length } 1/3^n \end{array} \right)$$

<u>Definition</u> The <u>middle third Cantor set</u> is $C = \bigcap_{n=0}^{\infty} C_n$

(= the set of points which lie in all of the C_n, i.e. never lie in the open middle third of any sub-interval)

Observe that the total "length" of C_n is $2^n \times (1/3)^n = (2/3)^n$

which $\to 0$ as $n \to \infty$,

so C has "length" equal to 0

A more general notion: $\{\varphi(c) : c \in C\}$

$$\{\varphi(c) : c \in C\} = \varphi(C)$$

<u>Defn</u>: A set of the form $\varphi(C)$ where C is the middle third Cantor set, and $\varphi : \mathbb{R} \to \mathbb{R}$ is a homeomorphism, is called a <u>Cantor set</u>.

Cantor sets K have the following properties (which could be regarded as <u>defining</u> properties of the notion of Cantor set):

1. K is "perfect", i.e. each point in K has other points in K which are arbitrarily close to it.

2. K is "totally disconnected", i.e. K contains no non-empty open intervals.

3. K is 'closed' (i.e. every sequence of points in K which converges to a limit l, is such that $l \in K$).

__Que__ How is this discussion related to dynamical systems?

__Defn__ For a map $f : [0,1] \to \mathbb{R}$, a point $x \in [0,1]$ is said to be __non-escaping__ if $f^n(x) \in [0,1]$ for all $n \geq 0$

(i.e. the orbit of x does not 'escape' from the domain of definition $[0,1]$).