Chaos & Fractals

Solutions 5

EXAM QUESTION: the questions below correspond to the various parts of Question 3 on the January 2023 exam paper

For parameters $\lambda > 0$, define $f_\lambda : \mathbb{R} \to \mathbb{R}$ by

$$f_\lambda(x) = \lambda x^2(1-x)\,.$$

Exercise 1. Show that there is a point $p \in \mathbb{R}$ which is a fixed point of f_λ for all $\lambda > 0$. Is p attracting or repelling? Justify your answer.

Clearly $f_\lambda(0) = 0$ for all λ, so $p = 0$ is a fixed point. Now $f'_\lambda(x) = \lambda(2x - 3x^2)$, so $f'_\lambda(0) = 0$. In particular $|f'_\lambda(0)| < 1$, therefore 0 is an attracting fixed point.

Exercise 2. Determine the value $\lambda_1 > 0$ such that f_λ has precisely one fixed point if $\lambda \in (0, \lambda_1)$, and precisely 3 fixed points if $\lambda > \lambda_1$. Justify your answer.

As noted in the solution to Exercise 1 above, the point $p = 0$ is a fixed point of f_λ for all λ. A point x is fixed by f_λ if and only if $\lambda x^2(1-x) = x$, and for $x \neq p = 0$ this holds if and only if $\lambda x(1-x) = 1$, if and only if

$$\lambda x^2 - \lambda x + 1 = 0\,.$$

This quadratic equation has solutions

$$x = \frac{1}{2\lambda}\left(\lambda \pm \sqrt{\lambda^2 - 4\lambda}\right)$$

which are real and distinct fixed points of f_λ if and only if $\lambda^2 - 4\lambda > 0$, and are non-real (hence not fixed points of $f_\lambda : \mathbb{R} \to \mathbb{R}$) if and only if $\lambda^2 - 4\lambda < 0$. Setting $\lambda_1 = 4$ we see that $\lambda^2 - 4\lambda < 0$ if $\lambda \in (0, \lambda_1)$, and $\lambda^2 - 4\lambda > 0$ if $\lambda > \lambda_1$. It follows that $\lambda_1 = 4$ has the required properties.

Exercise 3. For $\lambda > \lambda_1$, let $x_\lambda^- < x_\lambda^+$ denote the two fixed points of f_λ which are not equal to p. Determine explicit formulae for x_λ^- and x_λ^+ in terms of λ.

From the above we see that

$$x_\lambda^- = \frac{1}{2\lambda}\left(\lambda - \sqrt{\lambda^2 - 4\lambda}\right)\,, \qquad x_\lambda^+ = \frac{1}{2\lambda}\left(\lambda + \sqrt{\lambda^2 - 4\lambda}\right)\,.$$

Exercise 4. Show that x_λ^- is a repelling fixed point of f_λ for all $\lambda > \lambda_1$.

To prove that x_λ^- is repelling we will show that $f'_\lambda(x_\lambda^-) > 1$.
Now $f'_\lambda(x) = 2\lambda x - 3\lambda x^2$, so

$$f'_\lambda(x_\lambda^-) = 2\lambda x_\lambda^- - 3\lambda(x_\lambda^-)^2 = 2\lambda x_\lambda^- - 3(\lambda x_\lambda - 1) = 3 - \lambda x_\lambda^-$$

(where we used that $x = x_\lambda^-$ satisfies $\lambda x^2 - \lambda x + 1 = 0$).
It remains to show that $3 - \lambda x_\lambda^- > 1$, i.e. that

$$3 - \frac{1}{2}(\lambda - \sqrt{\lambda^2 - 4\lambda}) > 1\,.$$

For this, note that $\lambda > \lambda_1 = 4$, so $4\lambda > 16$, so $-4\lambda > -8\lambda + 16$, so

$$\lambda^2 - 4\lambda > \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2\,,$$

so $\sqrt{\lambda^2 - 4\lambda} > \lambda - 4$, so $4 > \lambda - \sqrt{\lambda^2 - 4\lambda}$, so $3 - \frac{1}{2}(\lambda - \sqrt{\lambda^2 - 4\lambda}) > 1$, as required.

Exercise 5. Determine the value $\lambda_2 > \lambda_1$ such that if $\lambda \in (\lambda_1, \lambda_2)$ then x_λ^+ is an attracting fixed point of f_λ, and if $\lambda > \lambda_2$ then x_λ^+ is a repelling fixed point of f_λ. Justify your answer.

By a calculation similar to the above we see that

$$f'_\lambda(x_\lambda^+) = 3 - \lambda x_\lambda^+ = 3 - \frac{1}{2}(\lambda + \sqrt{\lambda^2 - 4\lambda})\,,$$

which is decreasing in λ. The value λ_2 is such that $f'_{\lambda_2}(x_{\lambda_2}^+) = -1$, since then $|f'_\lambda(x_\lambda^+)| < 1$ for $\lambda \in (\lambda_1, \lambda_2)$ and $|f'_\lambda(x_\lambda^+)| > 1$ for $\lambda > \lambda_2$.
So λ_2 is the solution to the equation

$$3 - \frac{1}{2}(\lambda + \sqrt{\lambda^2 - 4\lambda}) = -1\,.$$

That is, $8 = \lambda + \sqrt{\lambda^2 - 4\lambda}$, i.e. $(8 - \lambda)^2 = \lambda^2 - 4\lambda$, i.e. $\lambda^2 - 16\lambda + 64 = \lambda^2 - 4\lambda$, i.e. $64 = 12\lambda$, i.e. $\lambda = 16/3$.
Therefore $\lambda_2 = 16/3$.

Exercise 6. Show that there exists $\lambda \in (5, 6)$ such that $2/3$ is a point of least period 2 for f_λ.

Now $f_\lambda(2/3) = 4\lambda/27$, and since $\lambda \in (5, 6)$ then $\lambda \neq 9/2$, so $2/3$ is not a fixed point. So $2/3$ has least period 2 if and only if $g(\lambda) := f_\lambda^2(2/3) - 2/3 = 0$. Now

$$g(\lambda) = \lambda(4\lambda/27)^2(1 - 4\lambda/27) - \frac{2}{3} = \frac{16\lambda^3(27 - 4\lambda)}{27^3} - \frac{2}{3}$$

is a continuous function of λ, with

$$g(5) = \frac{16 \times 125 \times 7}{27^3} - \frac{2}{3} = \frac{14000}{19683} - \frac{2}{3} > 0\,,$$

and

$$g(6) = \frac{16 \times 216 \times 3}{27^3} - \frac{2}{3} = \frac{128}{243} - \frac{2}{3} < 0\,,$$

so by the Intermediate Value Theorem there exists $\lambda \in (5, 6)$ such that $g(2/3) = 0$. This value of λ is such that $2/3$ is a point of least period 2 for f_λ.