Queen Mary University of London

Main Examination period 2025 – January – Semester A

MTH6107: Chaos & Fractals SOLUTIONS

Duration: 2 hours

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The exam is intended to be completed within **2 hours**.

You should attempt ALL questions. Marks available are shown next to the questions.

The exam is closed-book, and **no outside notes are allowed**. **Calculators are not permitted** in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: O. Jenkinson, C. Beck

Question 1 [30 marks].

(a) Suppose we are given a map $f : \mathbb{R} \to \mathbb{R}$.

(i) What does it mean to say that $x \in \mathbb{R}$ is a **fixed point** for f? [**1**]

(ii) What does it mean to say that $x \in \mathbb{R}$ is a **periodic point** for f? [**2**]

(iii) How is the **least period** of a periodic point defined? [**1**]

(iv) What does it mean to say that $x \in \mathbb{R}$ is an **eventually periodic point** for f? [**2**]

(b) Give a detailed description of the **Sharkovskii order**, and a statement of Sharkovskii's Theorem. [**6**]

(c) Order the integers from 1 to 20 inclusive using Sharkovskii's order. [**6**]

(d) Suppose the map $f : [0,1] \to [0,1]$ is defined by

$$f(x) = \begin{cases} 2x + 1/3 & \text{for } x \in [0, 1/3) \\ 3(1-x)/2 & \text{for } x \in [1/3, 1]\,. \end{cases}$$

(i) For this map f, determine all its fixed points. [**4**]

(ii) For this map f, determine an eventually periodic point which is not periodic. [**4**]

(iii) For this map f, determine whether there exists a point of least period 3 and determine whether there exists a point of least period 7, being careful to justify your answer. [**4**]

(a) (i) It means that $f(x) = x$.

(ii) It means that $f^n(x) = x$ for some $n \in \mathbb{N}$.

(iii) It is the smallest natural number n such that $f^n(x) = x$.

(iv) It means that for some $m \geq 0$, the point $f^m(x)$ is a periodic point.

(b) Sharkovskii's order $\prec$ of the natural numbers is given by:

$$1 \prec 2 \prec 2^2 \prec 2^3 \prec \cdots \prec 2^m \prec \cdots$$
$$\vdots$$
$$\cdots \prec 2^k(2n-1) \prec \cdots \prec 2^k \cdot 7 \prec 2^k \cdot 5 \prec 2^k \cdot 3 \prec \cdots$$
$$\vdots$$
$$\cdots \prec 2(2n-1) \prec \cdots \prec 2 \cdot 7 \prec 2 \cdot 5 \prec 2 \cdot 3 \prec \cdots$$
$$\cdots \prec 2n-1 \prec \cdots \prec 7 \prec 5 \prec 3\,.$$

Sharkovskii's Theorem then says that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, and has a periodic orbit of least period n, then it has a periodic orbit of least period m for all $m \prec n$.

(c) $1 \prec 2 \prec 4 \prec 8 \prec 16 \prec 20 \prec 12 \prec 18 \prec 14 \prec 10 \prec 6 \prec 19 \prec 17 \prec 15 \prec 13 \prec 11 \prec 9 \prec 7 \prec 5 \prec 3$

(d) (i) The unique fixed point is $3/5$. Note that there is no fixed point in $[0, 1/3)$ because the equation $2x + 1/3 = x$ has no solution in $[0, 1/3)$, and the only fixed point in $[1/3, 1]$ is a solution to $3(1-x)/2 = x$, namely $x = 3/5$.

(ii) The point $2/15$ is eventually periodic but not periodic: note that $f(2/15) = 4/15 + 1/3 = 9/15 = 3/5$, which is the fixed point.

(iii) The orbit $\{0, 1/3, 1\}$ has least period 3, so each of its members is a point of least period 3.

Since f is continuous, the existence of a point of least period 3 implies a point of period 7, by Sharkovskii's Theorem.

Question 2 [20 marks].

(a) Define what it means for $f : \mathbb{R} \to \mathbb{R}$ to be

  (i) a **diffeomorphism**, [**3**]

  (ii) **order reversing**. [**3**]

(b) Prove that an order reversing diffeomorphism $f : \mathbb{R} \to \mathbb{R}$ has precisely one fixed point. [**10**]

(c) Decide whether a diffeomorphism $f : \mathbb{R} \to \mathbb{R}$ with precisely one fixed point is necessarily order reversing, giving reasons for your answer. [**4**]

(a) (i) A diffeomorphism is defined (in this module) to be a bijection such that both f and f^{-1} are C^1 maps, i.e. they are differentiable with continuous derivative.

  (ii) It means that if $x < y$ then $f(x) > f(y)$.

(b) Existence: Note that $\lim_{x\to\infty} f(x) = -\infty$, and $\lim_{x\to-\infty} f(x) = \infty$. Let $\Phi(x) = f(x) - x$, so that $\lim_{x\to-\infty} \Phi(x) = +\infty$ and $\lim_{x\to\infty} \Phi(x) = -\infty$. By the intermediate value theorem there exists $c \in \mathbb{R}$ with $\Phi(c) = 0$, i.e. $f(c) = c$, so c is a fixed point.

Uniqueness: Suppose $f(c) = c$ and $f(d) = d$, with $c < d$, say. Then

$$c = f(c) > f(d) = d$$

since f reverses order; but this inequality contradicts the previous inequality $c < d$, so indeed there can be only one fixed point.

(c) This is False - for example $f(x) = 2x$ has a single fixed point, at 0, but is order preserving rather than order reversing.

Question 3 [27 marks].

Let $f_\mu : [0,1] \to [0,1]$ be the family of logistic maps, defined by $f_\mu(x) = \mu x(1-x)$ for parameters $\mu \in [0,4]$.

(a) For $\mu \in [0,1)$, show that 0 is an attracting fixed point of f_μ. **[3]**

(b) For $\mu \in [0,1)$, determine (without giving justification) the basin of attraction for the fixed point 0. **[2]**

(c) For $\mu \in (1,3)$, compute the value of the non-zero fixed point x_μ of f_μ. **[3]**

(d) Compute the multiplier of this fixed point x_μ, and determine the largest value μ_1 with the property that the fixed point x_μ is attracting for all $\mu \in (1,\mu_1)$. **[4]**

(e) For $\mu = 4$, determine the periodic orbit of f_4 which has least period 2. **[5]**

(f) Compute the multiplier of the period-2 orbit from (e), and determine whether or not the orbit is repelling. **[5]**

(g) Briefly define what is meant by a **period-doubling bifurcation**. **[3]**

(h) How is the **Feigenbaum constant** defined? **[2]**

(a) $f_\mu(0) = \mu 0(1-0) = 0$ so 0 is a fixed point.
Now $f_\mu'(x) = \mu - 2\mu x$, so $f_\mu'(0) = \mu \in [0,1)$ when $\mu \in [0,1)$, so since the multiplier is smaller than 1 in absolute value then the fixed point 0 is attracting.
(b) The basin of attraction is all of $[0,1]$.
(c) The non-zero fixed point satisfies $1 = \mu(1-x)$, so $x_\mu = (\mu-1)/\mu$.
(d) $f'(x_\mu) = \mu - 2\mu x_\mu = \mu - 2(\mu-1) = 2-\mu$, and this is strictly smaller than 1 in modulus if $\mu \in (1,3)$, so $\mu_1 = 3$.
(e) Period-2 points are solutions to $4 \cdot 4x(1-x)(1-4x(1-x)) = x$.
That is, $-x(4x-3)(16x^2-20x+5) = 0$
So the 2-cycle is $\{\frac{1}{8}(5+\sqrt{5}), \frac{1}{8}(5-\sqrt{5})\}$.
(f) The multiplier is $(4-(5+\sqrt{5}))(4-(5-\sqrt{5})) = -4$, so the 2-cycle is repelling.
(g) A **period-doubling bifurcation** is the event such as occurs at $\mu = \mu_1$ (and at general $\mu = \mu_n$), whereby a formerly attracting period-n orbit ceases to be attracting, and a new attracting period-$2n$ orbit is born.
(h) If we denote by (μ_n) the sequence of parameter values at which the period-doubling bifurcations occur, the Feigenbaum constant d_∞ can be defined by:

$$d_\infty = \lim_{n\to\infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n}.$$

Question 4 [23 marks]. Given an iterated function system defined by the two maps $\phi_1(x) = (x+1)/10$ and $\phi_2(x) = (x+4)/10$, define $\Phi(A) = \phi_1(A) \cup \phi_2(A)$, and let C_k denote $\Phi^k([0,1])$ for $k \geq 0$.

(a) Determine the sets C_1 and C_2. **[4]**

(b) If C_k is expressed as a disjoint union of N_k closed intervals, compute the number N_k. **[3]**

(c) What is the common length of each of the N_k closed intervals whose disjoint union equals C_k? **[3]**

(d) Compute the box dimension of $C = \cap_{k=0}^{\infty} C_k$, being careful to justify your answer. **[5]**

(e) Give a description of the members of C in terms of the digits of their decimal expansion. **[4]**

(f) If $f : C \to C$ is defined by $f(x) = 10x \pmod 1$ then find a point $x \in C$ which has least period 2 under f. Express this x as a fraction p/q. **[4]**

(a)

$$C_1 = \left[\frac{1}{10}, \frac{2}{10}\right] \cup \left[\frac{4}{10}, \frac{5}{10}\right] \, ,$$

$C_2 = [\frac{11}{100}, \frac{12}{100}] \cup [\frac{14}{100}, \frac{15}{100}] \cup [\frac{41}{100}, \frac{42}{100}] \cup [\frac{44}{100}, \frac{45}{100}]$

(b) $N_k = 2^k$ because $N_0 = 1$ and the recursive procedure doubles the number of intervals at each step.

(c) The common length is 10^{-k}, because the length of the closed intervals decreases by a factor of 10 at each step, and the length of $C_0 = [0,1]$ is 1.

(d) If $\varepsilon_k = 1/10^k$ then $N(\varepsilon_k) = 2^k$, so the box dimension equals

$$\lim_{k\to\infty} \frac{\log N(\varepsilon_k)}{-\log \varepsilon_k} = \lim_{k\to\infty} \frac{k \log 2}{k \log 10} = \frac{\log 2}{\log 10} \, .$$

(e) C consists of those numbers in $[0,1]$ with a decimal expansion whose digits all belong to $\{1,4\}$.

(f) Such x has decimal expansion either $0.141414\ldots$ or $0.414141\ldots$. The first of these is the solution x to the equation
$x = \phi_1(\phi_2(x)) = (\phi_2(x)+1)/10 = (1+(x+4)/10)/10 = 14/100 + x/100$, i.e. $x = 14/99$. The second is $41/99$.

End of Paper.