On (μ_0, μ_1), $\frac{\mu - 1}{\mu}$ is attracting

On (μ_1, μ_2), a 2-cycle is attracting

On (μ_2, μ_3), a 4-cycle is attracting

$\vdots$

On (μ_n, μ_{n+1}), a 2^n-cycle is attracting.

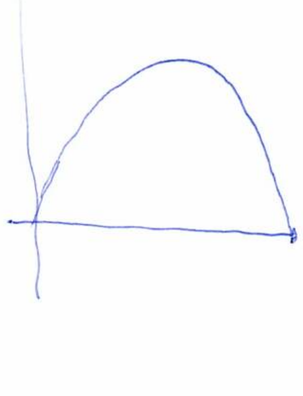
The lengths of these bifurcation parameter intervals are decreasing rapidly (in fact, at some geometric rate, i.e. exponentially fast)

We can write $d_k = \dfrac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k}$

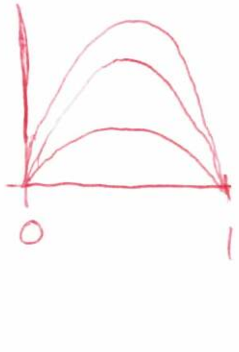
It was observed experimentally by Feigenbaum (and others) around 1975 that the sequence $(d_k)_{k=1}^{\infty}$ has a limit $\delta_\infty \approx 4.669202\ldots$

This is called the <u>Feigenbaum Constant</u> or <u>Feigenbaum ratio</u>.

It was also observed that we get the <u>same</u> "universal value" d_∞ for <u>all</u> parametrised families of maps <u>looking like</u> the logistic family



e.g. $g_\lambda(x) = \lambda \sin(\pi x)$



A natural question is : what happens for $\mu > \mu_\infty$?

All period -2^n orbits are repelling beyond the value μ_∞.

For some parameters μ there are other attracting periodic orbits whose period is <u>not</u> a power of 2.

For some parameters μ, the behaviour of f_μ is "chaotic".

<u>Example</u> $\mu = 4$: $f(x) = f_4(x) = 4x(1-x)$

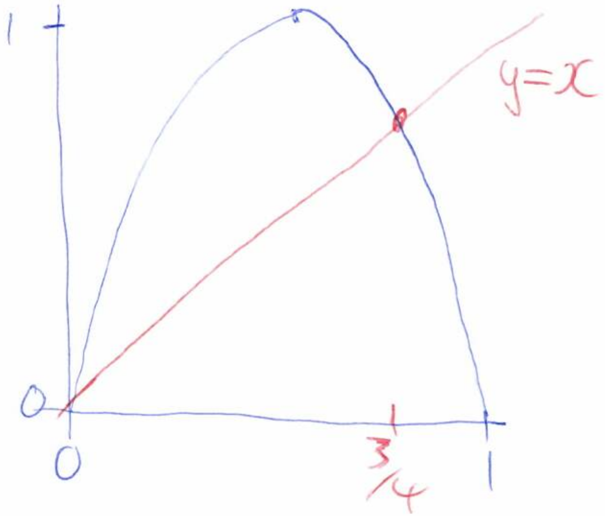
Fixed points : $x = 0$ and $x = 3/4$

$$\left(= \frac{\mu - 1}{\mu} \right)$$

Now $f'(x) = 4 - 8x$

So $|f'(x)| = \begin{cases} 4 & \text{if } x = 0 \\ |-2| = 2 & \text{if } x = 3/4 \end{cases}$

So both fixed points are repelling.



Period-2 points:

$$f^2(x) = x$$

$$\Longleftrightarrow f(f(x)) = x$$

$$\Longleftrightarrow f(4x(1-x)) = x$$

$$\Longleftrightarrow 4.(4x(1-x))(1 - 4x(1-x)) = x$$

$$\Longleftrightarrow -x(4x-3)(16x^2 - 20x + 5) = 0$$

So the roots of this quadratic

$p_\pm = \dfrac{5 \pm \sqrt{5}}{8}$ constitute a 2-cycle

This 2-cycle is <u>repelling</u>, because

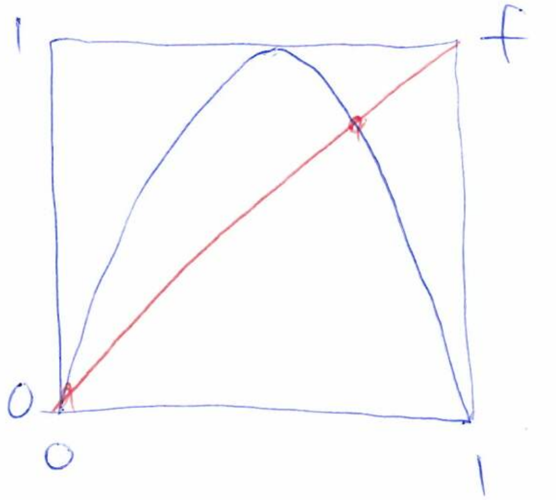
$|(f^2)'(p_+)| = |f'(p_+)f'(p_-)| = \ldots = |-4| = 4 > 1$

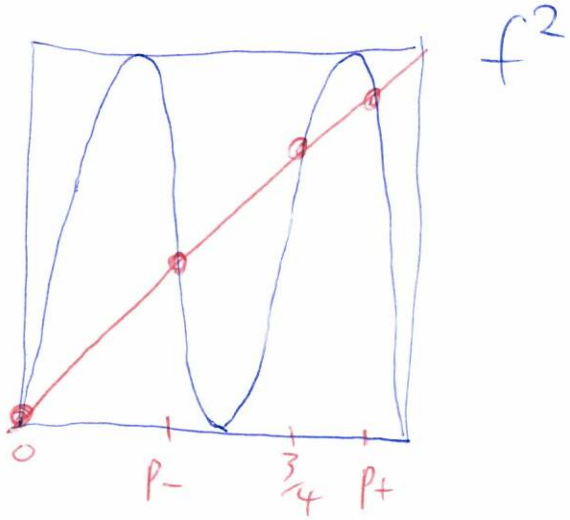
Period-3 points : $f^3(x) = x$

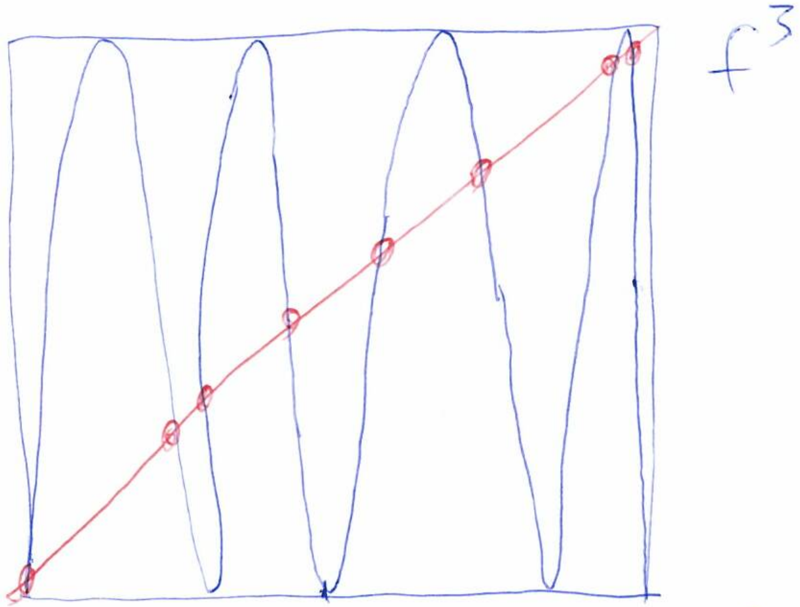
i.e. $f^3(x) - x$ } degree-8 polynomial

$\quad\ ||$

$x(4x-3)$ (cubic factor) (cubic factor)







There are two 3-cycles for f. Both 3-cycles are repelling. Therefore by Sharkovskii's theorem there are n-cycles for all $n \in \mathbb{N}$.

Topological conjugacy

Defn If X and Y are intervals in $\mathbb{R}$, we say $h: X \to Y$ is a <u>homeomorphism</u> if h is a bijection (i.e. invertible) and both h and h^{-1} are continuous.

Defn Let X and Y be intervals in $\mathbb{R}$. Assume we have maps $f: X \to X$ and $g: Y \to Y$. We say that $h: X \to Y$ is a <u>topological conjugacy</u> from f to g if

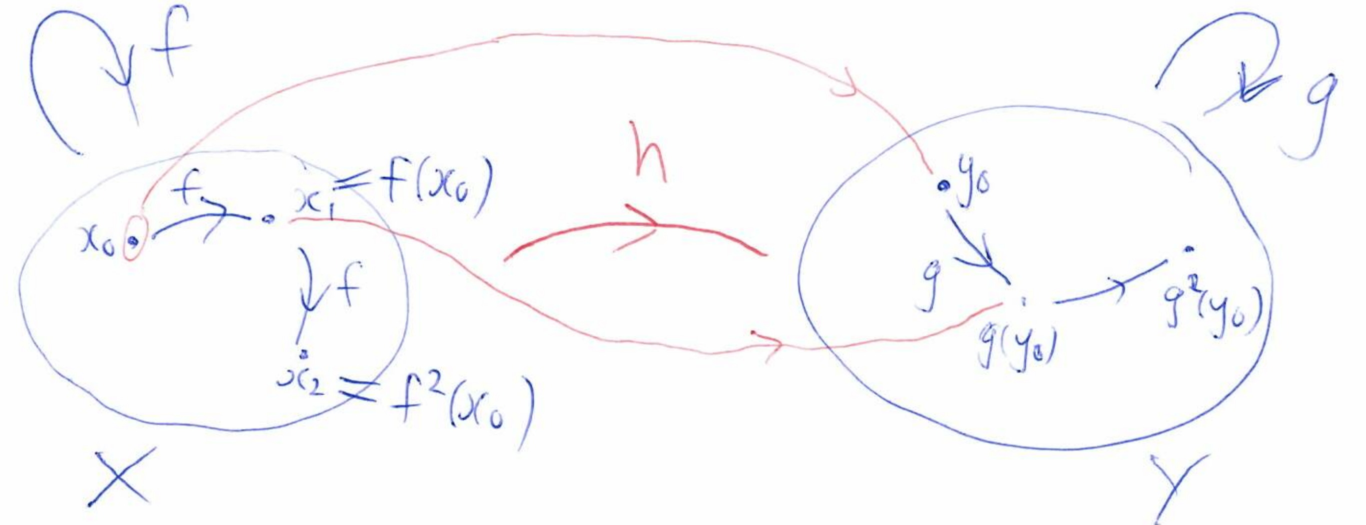
(i) h is a homeomorphism

(ii) $h \circ f = g \circ h$

Note regarding terminology:

- We also refer to h as a topological conjugacy between f and g or between g and f
- We also say that f and g are topologically conjugate
- We also say that f is topologically conjugate to g

Remarks (a) Conjugacy can be viewed as a change of coordinates/variables



So if we want to study the orbit of x_0 under f in X, we can use h to equivalently study the orbit of $y_0 = h(x_0)$ under g in Y.

(b) The <u>conjugacy equation</u>

$h \circ f = g \circ h$ can be written as

$$f = h^{-1} \circ g \circ h$$

or $$h \circ f \circ h^{-1} = g$$

or $$f \circ h^{-1} = h^{-1} \circ g$$

$$\left[\text{i.e. } h^{-1} \circ g = f \circ h^{-1} \right]$$

(c) Another way of thinking of the conjugacy equation (popular with algebraists) is that it is saying that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

<u>Example</u> <u>Claim</u>: We can find a topological conjugacy from the logistic map

$$f_\mu(x) = \mu x(1-x) \qquad (\mu > 0)$$

to $g_c(x) = x^2 + c$, for some suitable value of c (depending on μ).

To see this, we shall view f_μ and g_c as maps $\mathbb{R} \to \mathbb{R}$ (i.e. $X = Y = \mathbb{R}$)

Let's guess that the conjugacy h is of the form $h(x) = \alpha x + \beta$ $(\alpha \neq 0)$

(i.e. assume h is linear (affine))

Such an h is clearly a homeomorphism.

We want to find α, β such that
$$h(f_\mu(x)) = g_c(h(x))$$
for all $x \in \mathbb{R}$

i.e. $\alpha f_\mu(x) + \beta = g_c(\alpha x + \beta)$

i.e. $\alpha \mu x(1-x) + \beta = (\alpha x + \beta)^2 + c$

i.e. $\alpha \mu x - \alpha \mu x^2 + \beta = \alpha^2 x^2 + 2\alpha\beta x + \beta^2 + c$

Comparing coefficients gives:

x^2 : $-\alpha\mu = \alpha^2$ i.e. $\alpha = -\mu$

x : $\alpha\mu = 2\alpha\beta$ i.e. $\beta = \mu/2$

constant: $\beta = \beta^2 + c$ i.e. $c = \beta - \beta^2$

i.e. $c = \mu/2 - (\mu/2)^2$

So for $c = \frac{\mu}{2} - \left(\frac{\mu}{2}\right)^2$ then f_μ and g_c are indeed topologically conjugate, where the topological conjugacy is given by the homeomorphism

$$\underset{\displaystyle h_\mu(x)}{\overset{\displaystyle h(x)}{\|}} = -\mu x + \frac{\mu}{2}$$

e.g. So for example if $\mu = 4$, i.e. we are considering $f_4(x) = 4x(1-x)$ then this is topologically conjugate to $g_{-2}(x) = x^2 - 2$

because
$$\begin{aligned} c &= \frac{\mu}{2} - \left(\frac{\mu}{2}\right)^2 \\ &= \frac{4}{2} - \left(\frac{4}{2}\right)^2 \\ &= 2 - 4 = -2 \end{aligned}$$

[Recall that earlier we looked at $f(x) = 4x(1-x)$, and a few weeks ago we studied $x \longmapsto x^2 - 2$]

Lemma Topological conjugacy is an equivalence relation.

Proof: Exercise. $\square$

<u>Lemma</u> If $h: X \to Y$ is a topological conjugacy between $f: X \to X$ and $g: Y \to Y$ then h is also a topological conjugacy between f^n and g^n, for all $n \in \mathbb{N}$.

<u>Proof</u> We can write the conjugacy equation as $f = h^{-1} \circ g \circ h$.

Then $f^n = f \circ f \circ \ldots \circ f$

$$= (h^{-1} \circ g \circ h) \circ (h^{-1} \circ g \circ h) \circ \ldots \circ (h^{-1} \circ g \circ h)$$

these 'cancel' since $h \circ h^{-1}$ is the identity map (on Y)

$$= h^{-1} \circ g \circ id \circ g \circ id \circ g \ldots \circ id \circ g \circ h$$

$$= h^{-1} \circ g \circ g \circ \ldots \circ g \circ h$$

$$= h^{-1} \circ g^n \circ h$$ and this is a conjugacy equation, so f^n and g^n are topologically conjugate.

$\square$

Proposition Topological conjugacy preserves orbits, periodic points, periodic orbits, and (least) periods of orbits.

Proof Examine $\mathcal{O}_f(x_0) = \{f^n(x_0) : n \geq 0\}$ (the orbit of x_0 under f).

Then
$$h(\mathcal{O}_f(x_0)) = \{h(f^n(x_0)) : n \geq 0\}$$

by previous Lemma

$$= \{g^n(h(x_0)) : n \geq 0\}$$
$$= \mathcal{O}_g(h(x_0))$$

(the orbit of $y_0 = h(x_0)$ under g).

So we have seen that the image under h of an orbit is itself an orbit.

In particular, a periodic orbit for f is mapped by h to a periodic orbit for g.

__Exercise__: Check that the (least) period of the orbit is preserved by h.

__Important__ The preceding Proposition gives a way of showing that 2 maps f and g are <u>not</u> topologically conjugate.

__Corollary__ If for some $n \in \mathbb{N}$, the map f has a point of period n, but g does not have a point of period n, then f and g are not topologically conjugate.

__Example__: If f has a fixed point but g does not, then f and g are not topologically conjugate.

__Corollary__: If two maps f, g are topologically conjugate, then for every $n \in \mathbb{N}$ the number of n-cycles for f is equal to the number of n-cycles for g.

<u>Example</u> For $\mu = 1 + \sqrt{8}$, the logistic map $f_\mu(x) = \mu x(1-x)$ is topologically conjugate to

$$g_c(x) = x^2 + c$$

where $c = \mu/2 - (\mu/2)^2$

$$= \tfrac{1}{2}(1+\sqrt{8}) - \tfrac{1}{4}(1+\sqrt{8})^2$$

$$= \tfrac{1}{2} + \tfrac{1}{2}\sqrt{8} - \tfrac{1}{4}(1 + 2\sqrt{8} + 8)$$

$$= -7/4$$

It is convenient to study $g_c(x) = x^2 + c$ with $c = -7/4$, in order to show the 'birth' of a period-3 orbit at this value of c.

The equation $g_c^3(x) = x$

becomes

$$((x^2+c)^2+c)^2 + c = x \qquad (*)$$

Now the (two) fixed points of g_c satisfy $(*)$, in other words

$$g_c(x) - x = x^2 - x + c$$

is a factor of $g_c^3(x) - x$.

Factorising $g_c^3(x) - x$, we get that $(*)$ becomes:

$$(x^2 - x + c)\underbrace{(x^6 + x^5 + (3c+1)x^4 + \dots)}_{} = 0 \qquad (**)$$

For <u>general</u> c, we expect complex (not real) roots, however when $c = -7/4$ we find that the LHS of $(**)$ is

$$(x^2 - x + \underset{-7/4}{c})\left(x^3 - \frac{x^2}{2} - \frac{9x}{4} - \frac{1}{8}\right)^2$$

The roots of this cubic are real and give a 3-cycle