Fixed point(s) given by $f(x) = x$.

Clearly $x + 1/2 = x$ has <u>no solutions</u>, so it suffices to consider

$$2 - 2x = x$$

i.e. $2 = 3x$

i.e. $x = 2/3$.

Period-2 orbits are given by solutions to the equation $f^2(x) = x$.

Assume $x \in [0, 1/2]$. Solve $f^2(x) = x$.

i.e. $f(f(x)) = x$

since $x \in [0, 1/2]$

i.e. $f(\underbrace{x + 1/2}_{\in [1/2, 1]}) = x$

i.e. $2 - 2(x + 1/2) = x$

i.e. $2 - 2x - 1 = x$

i.e. $1 = 3x$

So $x = 1/3$ has period 2 (indeed it has <u>least</u> period 2)

Therefore $f(1/3) = 1/3 + 1/2 = 5/6$ is <u>also</u> of period 2

[Check this: $f(5/6) = 2 - 2(5/6) = 2 - 5/3 = 1/3$]

So $\{1/3, 5/6\}$ is a 2-cycle.

There is also a 3-cycle, namely $\{0, 1/2, 1\}$

$$\begin{tikzcd} 0 \arrow[r, mapsto, "f"] & \frac{1}{2} \arrow[r, mapsto, "f"] & 1 \arrow[ll, bend left, "f"] \end{tikzcd}$$

Therefore, by Sharkovskii's Theorem, this f has n-cycles for <u>all</u> $n \in \mathbb{N}$

Logistic family of maps

Definition The <u>logistic family</u> is the family of functions f_μ defined by

$$f_\mu(x) = \mu x(1-x)$$

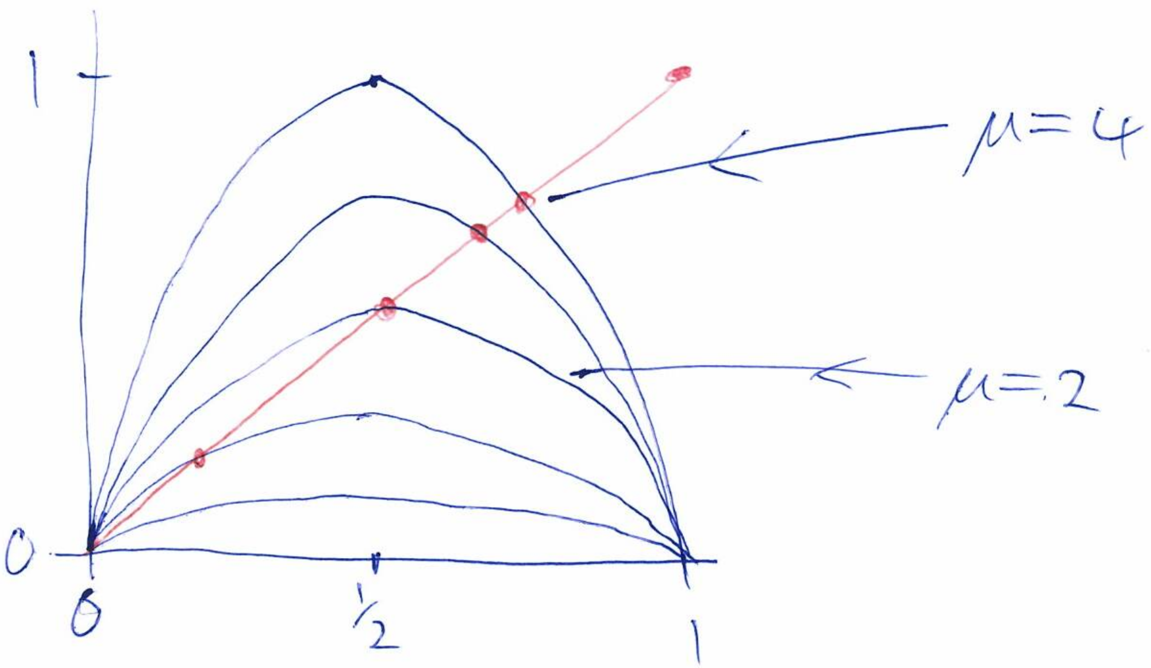
where $\mu > 0$ is a parameter.

As usual, we will study the dynamical system given by f_μ

i.e. $x_{n+1} = \mu x_n(1-x_n$

- First studied by R. May (1976) to model insect population dynamics
- Today we treat this system as a model which has a <u>transition to chaos</u>

By convention we shall assume $x \in [0,1]$ (i.e. we consider $f_\mu : [0,1] \to [0,1]$, and $\mu \in [0,4]$)



Graphs of f_μ for various parameters μ

<u>Derivative</u>: $f_\mu'(x) = \mu - 2\mu x$

The critical point (a maximum) is at $x = \frac{1}{2}$ for <u>all</u> parameters μ.

The maximum <u>value</u> of f_μ is then

$$f_\mu(\tfrac{1}{2}) = \mu/4.$$

<u>Note</u> $f_\mu(x) = f_\mu(1-x)$ for all x, so the graph of f_μ is symmetric about the point $\frac{1}{2}$.

<u>Fixed points</u> : $f_\mu(x) = x$

i.e. $\mu x(1-x) = x$

i.e. $0 = \mu x^2 + (1-\mu)x$

i.e. $x = 0$ or $0 = \mu x + (1-\mu)$

i.e. $x = \frac{\mu-1}{\mu} = 1 - \frac{1}{\mu}$

So $x=0$ and $x = \frac{\mu-1}{\mu}$ are the fixed points of f_μ, though note that if $\mu < 1$ then the "fixed point" $\frac{\mu-1}{\mu}$ is <u>negative</u> therefore outside of $[0,1]$, so we do not consider it (since we are thinking of f_μ as a map $[0,1] \to [0,1]$).

When are these fixed points attracting?

We first calculate $|f_\mu'(x)|$

$$|f_\mu'(x)| = |\mu - 2\mu x|$$

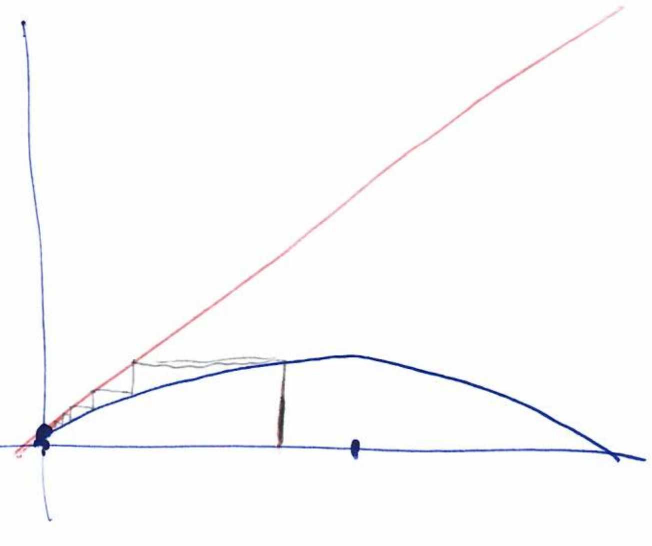
$$= \begin{cases} \mu & \text{if } x = 0 \\ |2-\mu| & \text{if } x = \frac{\mu-1}{\mu} \end{cases}$$

So for each fixed point we can (using a previous Theorem) say:

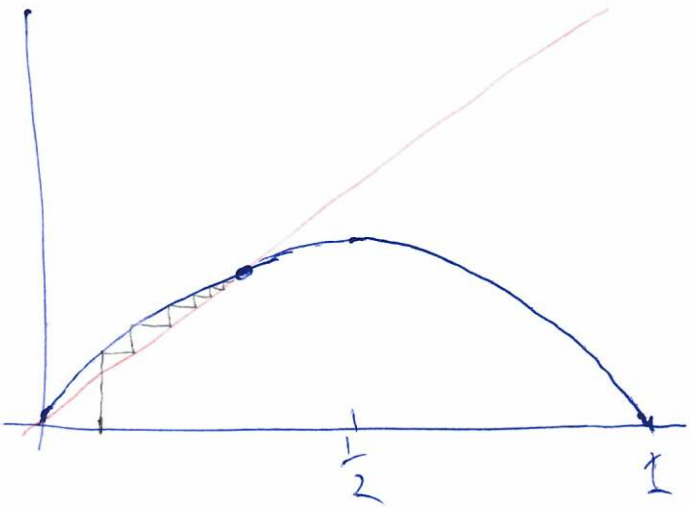
- The fixed point 0 is attracting if $0 \leq \mu < 1$, and repelling if $1 < \mu \leq 4$
- The fixed point at $\frac{\mu-1}{\mu}$ is attracting if $|2-\mu| < 1$, i.e. if $1 < \mu < 3$ and is repelling if $3 < \mu \leq 4$

<u>Remark</u> So $\mu = 1$ is a key transition parameter, since the fixed point 0 stops being attracting, and the 'new' fixed point $\frac{\mu-1}{\mu}$ is 'born'.

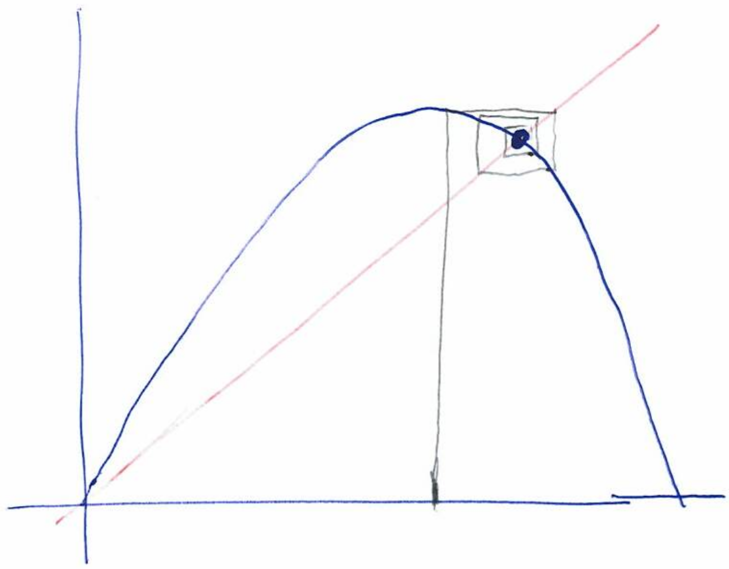
<u>Graph of f_μ</u> (for $0 \leq \mu < 1$)



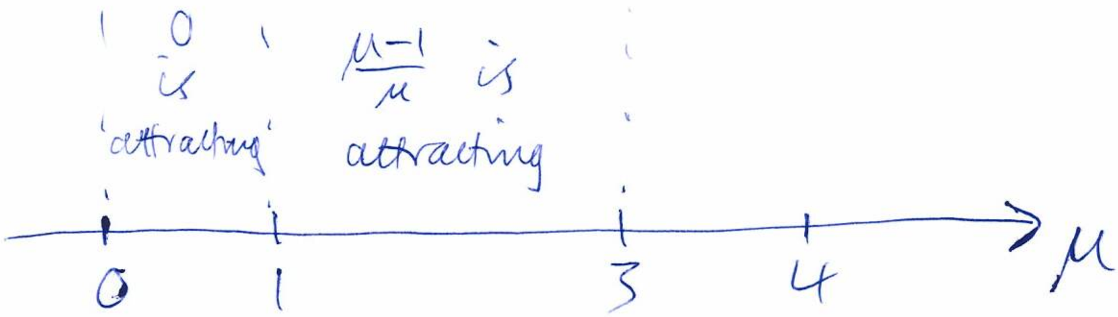
Graph of f_μ (for $1 < \mu < 2$)



Graph of f_μ (for $2 < \mu < 3$)



So far we can summarise as follows:



For $\mu > 3$, both fixed points 0 and $\frac{\mu-1}{\mu}$ are repelling, but what is happening dynamically? We will see that $\mu = 3$ marks the 'birth' of a period-2 orbit, and for $\mu > 3$ which "are not too much bigger than 3" this period-2 orbit is attracting.

Let's investigate period-2 points in detail.

Period-2 ~~orbits~~ points are solutions to the equation $f_\mu^2(x) = x$

i.e. $f_\mu(f_\mu(x)) = x$

i.e. $f_\mu(\mu x(1-x)) = x$

i.e. $f_\mu(\mu x(1-x)) - x = 0$

i.e. $\mu\Big(\mu x(1-x)\Big)\Big(1 - \mu x(1-x)\Big) - x = 0$

i.e. $-x + (\mu^2 x - \mu^2 x^2)(1 - \mu x + \mu x^2) = 0$

check this

i.e. $x(-\mu x + \mu - 1)\left(\mu^2 x^2 - (\mu^2+\mu)x + (\mu+1)\right) = 0$

Note that this is $f_\mu(x) - x$, i.e. solutions are fixed points

The points of least period 2 will be roots of this quadratic

The points of least period 2 are the roots of $\mu^2 x^2 - (\mu^2+\mu)x + \mu+1$

i.e. The two points, p_+ and p_-, of least period 2 are

$$p_\pm = \frac{1}{2\mu^2}\left(\mu^2+\mu \pm \sqrt{(\mu^2+\mu)^2 - 4\mu^2(\mu+1)}\right)$$

$$= \frac{1}{2\mu}\left(\mu+1 \pm \sqrt{(\mu+1)(\mu-3)}\right)$$

Notice that if $\mu=3$ then both p_+ and p_- are equal to the fixed point $\frac{\mu-1}{\mu} = 2/3 = 1/6\,(3+1 \pm \sqrt{0})$
$= \frac{1}{2\mu}(\mu+1 \pm \sqrt{0})$

If $\mu<3$ then $p_\pm$ do not exist (we do not consider non-real solutions)

For which values of μ ($\in [3,4]$) is the 2-cycle $\{p_+, p_-\}$ attracting or repelling?

We will analyse the modulus of the multiplier of the 2-cycle.

Let
$$M = \left| (f_\mu^2)'(p_+) \right|$$
$$= \left| f_\mu'(f_\mu(p_+)) \cdot f_\mu'(p_+) \right|$$
$$= \left| f_\mu'(p_-) \cdot f_\mu'(p_+) \right|$$
$$= \left| (\mu - 2\mu p_-) \cdot (\mu - 2\mu p_+) \right|$$
$$= \left| \mu^2 - 2\mu^2 (p_- + p_+) + 4\mu^2 p_- p_+ \right|$$

$\not\to$ check!!

$$= |\mu^2 - 2\mu(\mu+1) + 4(\mu+1)|$$

$$= |4 + 2\mu - \mu^2|$$

Recall that the 2-cycle $\{p_+, p_-\}$ is attracting if $M < 1$ and repelling if $M > 1$.

Now $M = 1$ when

either ① $4 + 2\mu - \mu^2 = 1$

or ② $4 + 2\mu - \mu^2 = -1$

Case ① : $4 + 2\mu - \mu^2 = 1$

$$\iff 0 = \mu^2 - 2\mu - 3$$

$$= (\mu - 3)(\mu + 1)$$

So $M = 1$ when $\mu = 3$ (we ignore the case $\mu = -1$ since we only consider $\mu \in [0,4]$)

Case (2) : $4 + 2\mu - \mu^2 = -1$

$\iff \mu^2 - 2\mu - 5 = 0$

$\iff \mu = \frac{1}{2}\left(2 \pm \sqrt{4 + 20}\right)$

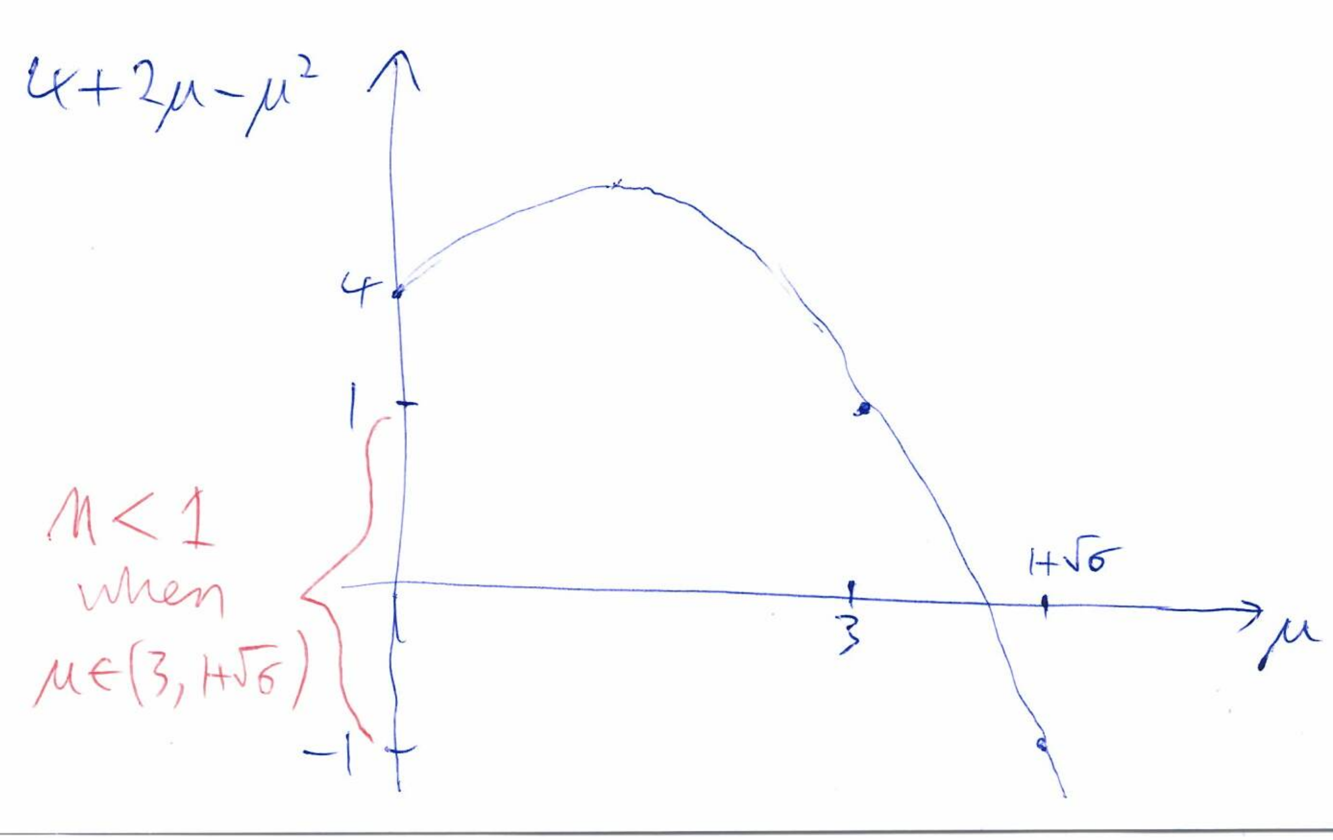
$= \frac{1}{2}\left(2 \pm 2\sqrt{6}\right)$

$= 1 \pm \sqrt{6}$

So $M = 1$ when $\mu = 1 + \sqrt{6}$

$\approx 3.449...$

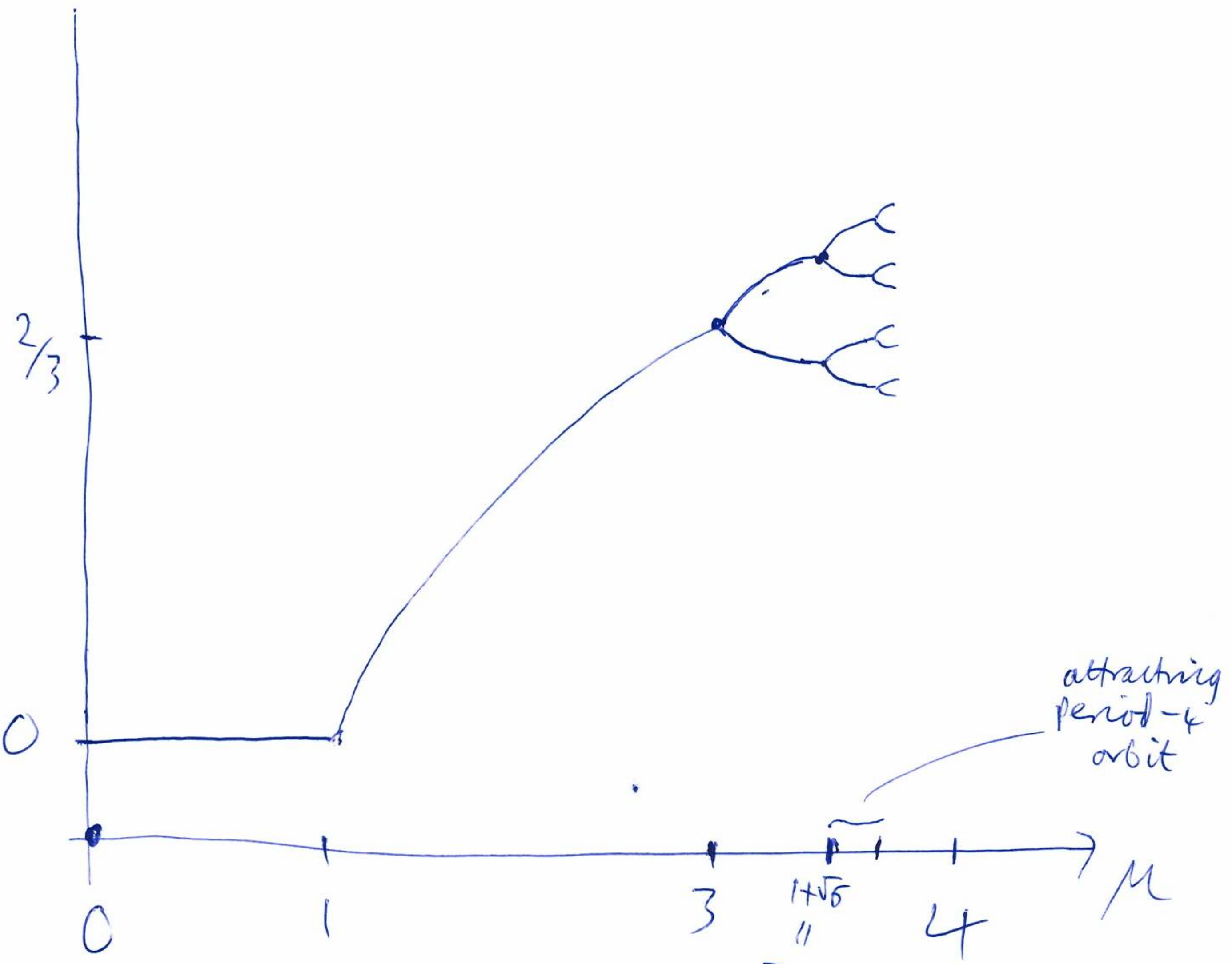
(we ignore the value $1 - \sqrt{6}$, since it is negative)



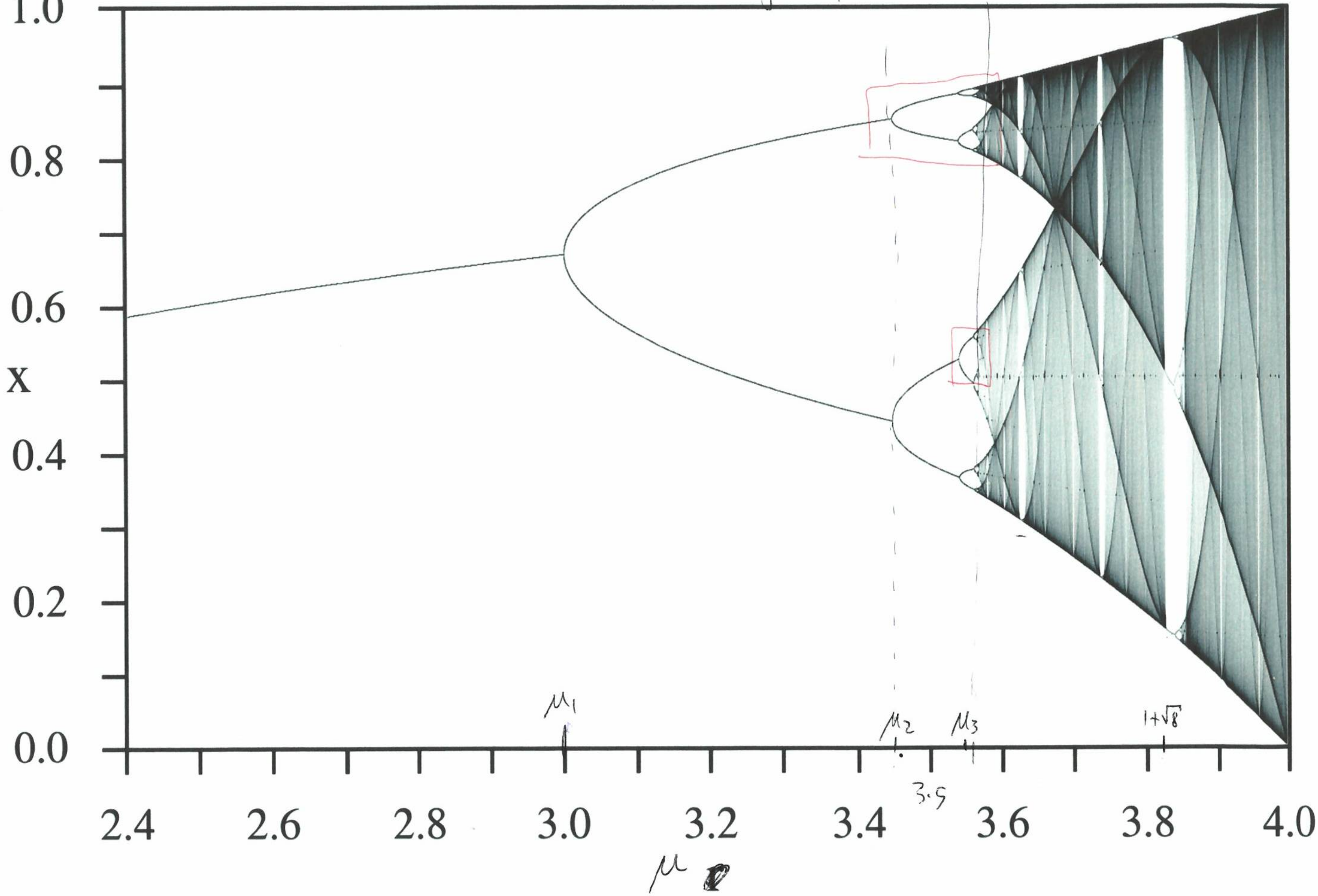
So $M < 1$ for $3 < \mu < 1+\sqrt{6}$

i.e. The 2-cycle $\{p_+, p_-\}$
is attracting if $3 < \mu < \underset{\substack{\shortparallel \\ 3.449\ldots}}{1+\sqrt{6}}$

The evolution of the attracting periodic orbit, as μ increases, can be summarised by the so-called <u>bifurcation diagram</u>



Bifurcation Diagram



It turns out that at each stage, the loss of attractingness / attraction of the period-2^n orbit gives rise to an attracting periodic orbit of least period 2^{n+1}.

This is called a <u>period-doubling bifurcation</u>.

In fact, since we have many such period-doubling bifurcations, this is sometimes referred to as a <u>period-doubling cascade</u>.

If we let μ_n denote the special values of μ at which these period-doubling bifurcations occur, then:

$$(\mu_0 = 1)$$

$$\mu_1 = 3$$

$$\mu_2 = 1+\sqrt{6} \simeq 3.449$$

$$\mu_3 = 3.544\ldots$$

$$\mu_4 = 3.564\ldots$$

$$\mu_5 = 3.568\ldots$$

$$\vdots$$

$$\mu_\infty = 3.569946\ldots = \lim_{n\to\infty} \mu_n$$

The sequence $(\mu_n)_{n=0}^{\infty}$ converges to some value μ_∞, as $n \to \infty$.