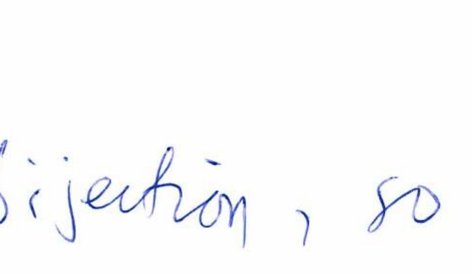
<u>Proposition</u> If $f: \mathbb{R} \to \mathbb{R}$ is an order-reversing diffeomorphism then f has exactly one fixed point.

<u>Proof</u> First we show that f has a fixed point. Since f is order-reversing, we know that if $a < b$ then $f(a) > f(b)$.

Now $f: \mathbb{R} \to \mathbb{R}$ is a bijection, so

$$\lim_{x \to \infty} f(x) = -\infty$$



and $$\lim_{x \to -\infty} f(x) = +\infty$$

Now consider the function

$$g(x) := f(x) - x, \text{ so that}$$

$$\lim_{x \to \infty} g(x) = -\infty \quad \text{and} \quad \lim_{x \to -\infty} g(x) = +\infty$$

By the Intermediate Value Theorem, there exists $p \in \mathbb{R}$ such that $g(p) = 0$,
i.e. $f(p) = p$, i.e. p is a fixed point for f.

We next show that p is the <u>unique</u> fixed point.

Suppose (in order to obtain a contradiction) that there is some other fixed point, i.e. some value q, where $q \neq p$, such that $f(q) = q$.

Without loss of generality, suppose that

$$q < p. \quad (*)$$

Since f is order-reversing we have

$$f(q) > f(p) \quad (**)$$

But $f(q) = q$ and $f(p) = p$, so $(**)$ in fact says that $q > p$, which contradicts $(*)$.

This is a contradiction, so indeed there is a unique fixed point. $\square$

Note that for order-preserving diffeos we cannot make any general statements about the number of fixed points. Let us look at some examples:

<u>Example</u> $f(x) = x$

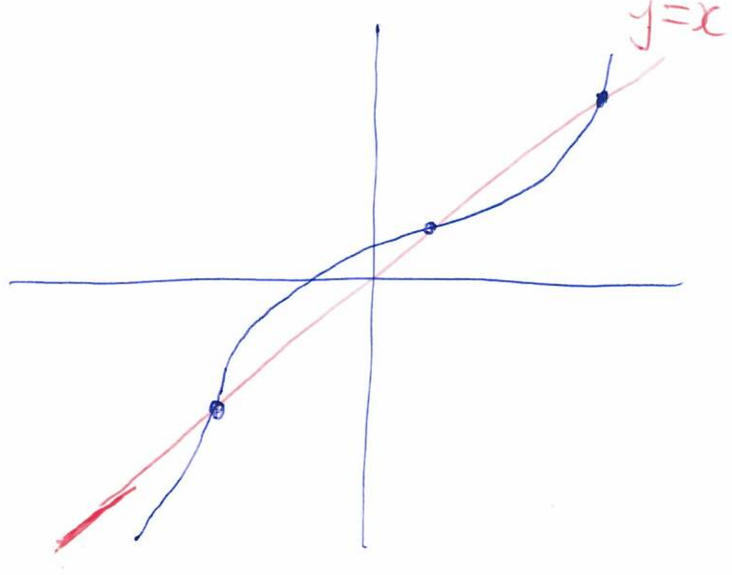
There <u>every</u> real number is a fixed point.

<u>Example</u> $f(x) = x + c$, where $c \neq 0$

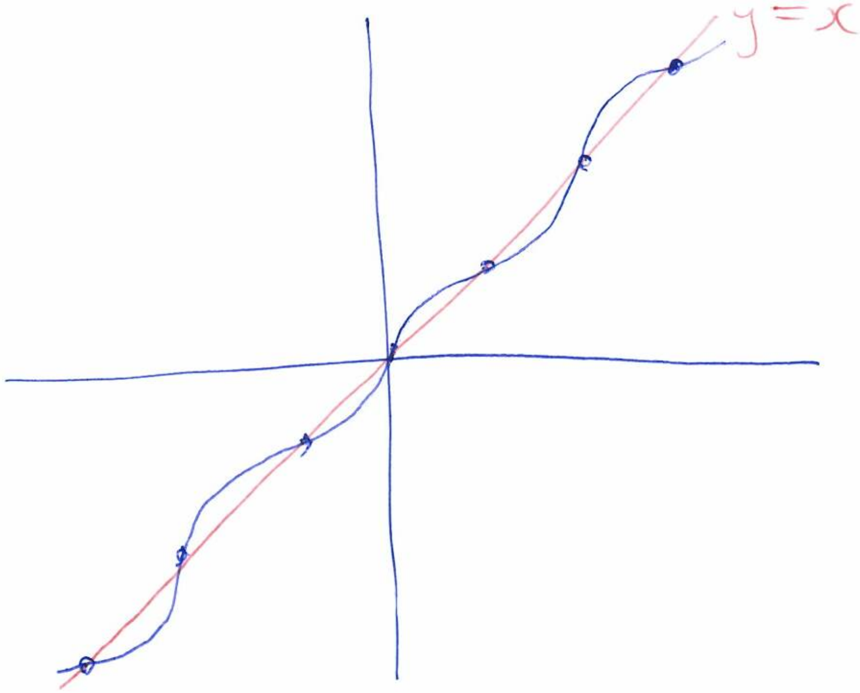
Here there are <u>no</u> fixed points.

<u>Example</u>

Illustration of an order-preserving diffeo with precisely 3 fixed points



Example $\quad f(x) = x + \frac{1}{2}\sin(x)$



The fixed point equation is $f(x) = x$

i.e. $x + \frac{1}{2}\sin(x) = x$

i.e. $\frac{1}{2}\sin(x) = 0$

So $x = n\pi$, $n \in \mathbb{Z}$, are the fixed points (infinitely many of them)

Note that $f'(x) = 1 + \frac{1}{2}\cos(x)$

$\geqslant 1 + \frac{1}{2}(-1)$

$= \frac{1}{2} > 0$ for all $x \in \mathbb{R}$

So f is an order-preserving diffeomorphism.

~~Note~~

We can check if the fixed points at $x = n\pi$, $n \in \mathbb{Z}$, are attracting or repelling:

$$f'(n\pi) = 1 + \tfrac{1}{2}\cos(n\pi)$$
$$= \begin{cases} 3/2 & \text{if } n \text{ is even} \\ 1/2 & \text{if } n \text{ is odd} \end{cases}$$

So (by the Theorem proved last week), the fixed point $n\pi$ is repelling if n is even (since $|f'(n\pi)| > 1$) and attracting if n is odd (since $|f'(n\pi)| < 1$).

Periodic points of higher period

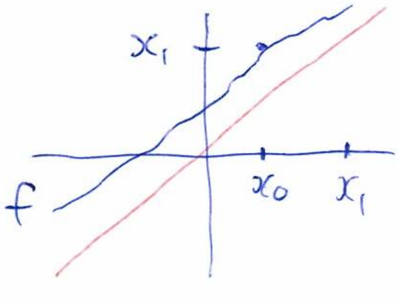
Proposition If $f: \mathbb{R} \to \mathbb{R}$ is an order-preserving diffeomorphism then any periodic point for f must be a fixed point.

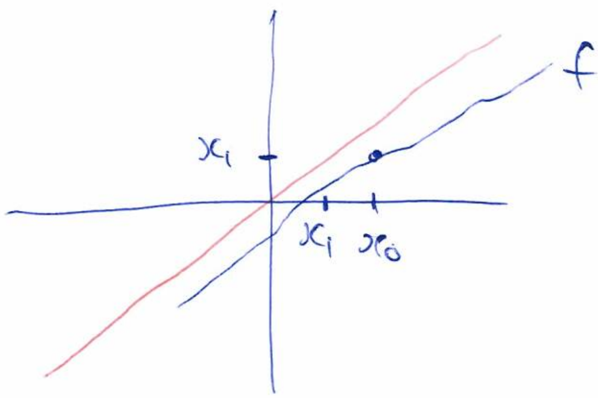
Proof Let $x_0 \in \mathbb{R}$. As usual, let

$$x_n = f^n(x_0).$$

If x_0 is not a fixed point, then

either $x_1 > x_0$ or $x_1 < x_0$.





If $x_0 < x_1$ then $f(x_0) < f(x_1)$,

i.e. $x_1 < x_2$

But then apply f again to get $f(x_1) < f(x_2)$,

i.e. $x_2 < x_3$

So $x_0 < x_1 < x_2 < x_3 < \ldots$

So in particular $x_k \neq x_0$ for all $k \geq 1$, therefore x_0 is not periodic.

Similarly, if $x_0 > x_1$ then the order-preserving property can be used to show that

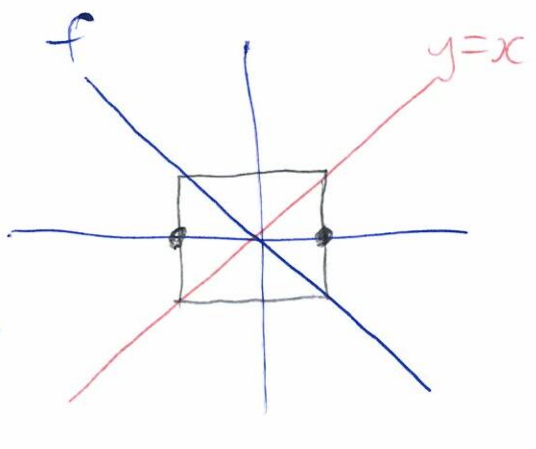
$$x_0 > x_1 > x_2 > x_3 > \ldots$$

and again we see that x_0 is not periodic.

$\square$

What about periodic points for order-reversing diffeomorphisms?

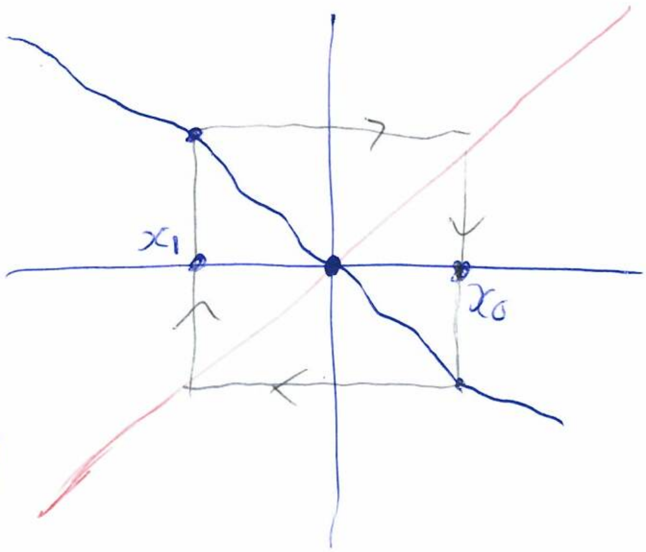
Example $f(x) = -x$



Here, 0 is the unique fixed point, but every other point has least period 2.

Example

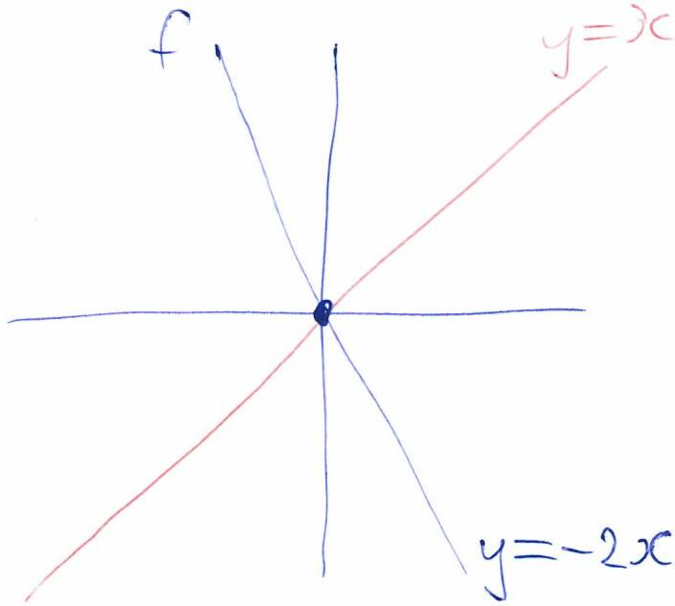
Illustration of another order-reversing diffeo with a 2-cycle $\{x_0, x_1\}$



Example

$f(x) = -2x$

Here there are <u>no</u> points of least period 2.



As we see with these examples, for order-reversing diffeos there <u>could</u> be points of least period 2, but there need not be.

Rather like the situation for fixed points of order-<u>preserving</u> diffeos, we will <u>not</u> prove any general result about least period 2 points for order-reversing diffeos.

However, there is an important result about points of least period > 2:

<u>Proposition</u> If $f: \mathbb{R} \to \mathbb{R}$ is an order-reversing diffeo then there are <u>no</u> points of least period <u>strictly</u> larger than 2.

<u>Proof</u> Since f is a diffeomorphism, so is $f^2 = f \circ f$. Notice that f^2 is order-<u>preserving</u> because

$$(f^2)'(x) = \underbrace{f'(f(x))}_{<0} \cdot \underbrace{f'(x)}_{<0}$$

which is > 0 (since $f' < 0$, and the product of 2 negative values is positive)

So by the previous Proposition, the order-preserving diffeo f^2 does not have any periodic points of least period > 1.

So f ~~is~~ itself does not have any periodic points of period $2m$ for $m > 1$, i.e. f has no points of least period k for even numbers $k > 2$.

To address the case where k is odd (and ≥ 3) we note that for such k, the map f^k is an order-reversing diffeomorphism.

But we have seen that order-reversing diffeomorphisms have precisely one fixed point, and of course the unique fixed point p for f is also a fixed point for f^k; therefore p is the unique fixed point for f^k.

So f^k has no other fixed points.
In other words, there are no period-k points for f (except for the fixed point p).
i.e. There are no points of least period k for f. $\square$

Summarising the situation for all diffeomorphisms:

	Fixed Points	Least period 2	Least period > 2
f order-preserving	Arbitrary number	None	None
f order-reversing	Exactly one	Arbitrary number	None

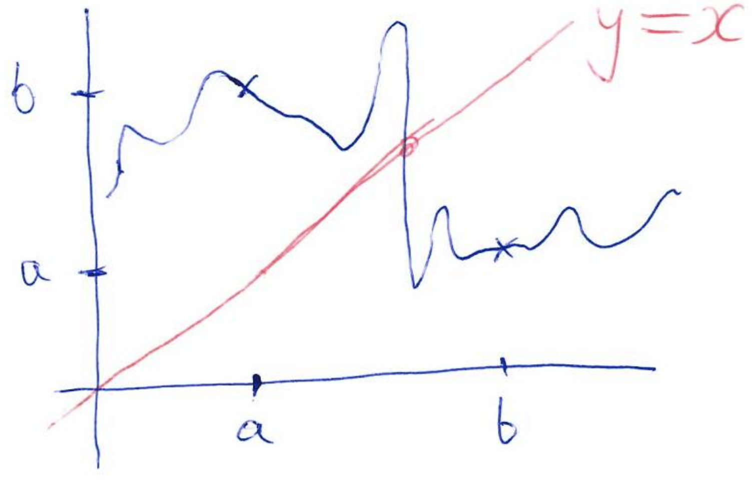
Dynamics of continuous maps f

- fixed points and periodic points

Proposition Let $f: \mathbb{R} \to \mathbb{R}$ be continuous. If f has an orbit of least period 2 then it has a fixed point.

Proof Let $\{a, b\}$ be a 2-cycle for f, with $a < b$.

Then $f(a) = b$, and $f(b) = a$.



Let $g(x) := f(x) - x$, so that a zero of g (i.e. a 'root' of g) is a fixed point for f.

Clearly f is continuous on $[a, b]$ since it is continuous on $\mathbb{R}$, therefore g is also continuous on $[a, b]$.

Also,
$$g(a) = f(a) - a = b - a > 0,$$
and
$$g(b) = f(b) - b = a - b < 0.$$

By the Intermediate Value Theorem (applied to the continuous function g) there exists $c \in (a, b)$ such that $g(c) = 0$, i.e. such that $f(c) = c$.

So f has a fixed point c. $\square$

Remark This is an example of "forcing", i.e. the presence of a periodic orbit of some period forces the presence of an orbit of another period.

[Here: "(Least) Period 2 $\Longrightarrow$ "Period 1"]

Perhaps the most famous result of this type is the "Period 3 implies chaos" (Li - Yorke) theorem, which more precisely states:

Theorem If $f: \mathbb{R} \to \mathbb{R}$ is continuous and has an orbit of least period 3, then it has periodic orbits of all other least periods n, for $n \in \mathbb{N}$.

This 'period-3 implies chaos' theorem is actually a consequence of a more general theorem due to Sharkovskii.

<u>Definition</u> The <u>Sharkovskii order</u> of the natural numbers $\mathbb{N}$ is given by:

$$1 \lhd 2 \lhd 4 \lhd 8 \lhd 16 \lhd \ldots \lhd 2^n \lhd \ldots$$

$$\vdots$$

$$\ldots \lhd 2^k(2n+1) \lhd \ldots\ldots \lhd 2^k \times 7 \lhd 2^k \times 5 \lhd 2^k \times 3$$

$$\vdots$$

$$\ldots \lhd 2^2 \times (2n+1) \lhd \ldots \quad \ldots \lhd 2^2 \times 7 \lhd 2^2 \times 5 \lhd 2^2 \times 3$$

$$\ldots \lhd 2(2n+1) \lhd \ldots \lhd 22 \lhd 18 \lhd 14 \lhd 10 \lhd 6$$

$$\ldots \lhd (2n+1) \lhd \ldots \lhd 11 \lhd 9 \lhd 7 \lhd 5 \lhd 3$$

Sharkovskii's Theorem Let $I=(a,b)\subseteq\mathbb{R}$ (we allow $I=\mathbb{R}=(-\infty,\infty)$)

Suppose $f:I\to I$ is continuous, and has an m-cycle (i.e. a periodic orbit of least period m).

Then f has an n-cycle for all those natural numbers n for which $n\lhd m$ (where $\lhd$ is the Sharkovskii order on $\mathbb{N}$).

Example Suppose $f:\mathbb{R}\to\mathbb{R}$ is continuous and has an orbit of least period $8=2^3$. Then by Sharkovskii's Theorem, f has a fixed point, and a 2-cycle, and a 4-cycle.

Example Suppose $f:\mathbb{R}\to\mathbb{R}$ is a continuous function, and has a 10-cycle. Then by Sharkovskii's Theorem, f has n-cycles for all even numbers n except $n=6$.

Example Let $f: [0,1] \to [0,1]$ be defined by

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{if } x \leq \frac{1}{2} \\ 2 - 2x & \text{if } x > \frac{1}{2} \end{cases}$$