Chaos & Fractals

Solutions 2

Exercise 1. Order the integers from 1 to 50 inclusive using Sharkovskii's ordering.

$1 \lhd 2 \lhd 4 \lhd 8 \lhd 16 \lhd 32 \lhd 48 \lhd 40 \lhd 24 \lhd 44 \lhd 36 \lhd 28 \lhd 20 \lhd 12 \lhd 50 \lhd 46 \lhd 42 \lhd 38 \lhd 34 \lhd 30 \lhd 26 \lhd 22 \lhd 18 \lhd 14 \lhd 10 \lhd 6 \lhd 49 \lhd 47 \lhd 45 \lhd 43 \lhd 41 \lhd 39 \lhd 37 \lhd 35 \lhd 33 \lhd 31 \lhd 29 \lhd 27 \lhd 25 \lhd 23 \lhd 21 \lhd 19 \lhd 17 \lhd 15 \lhd 13 \lhd 11 \lhd 9 \lhd 7 \lhd 5 \lhd 3$

Exercise 2. For the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x-1)(1-3x^2/2)$, determine the orbit of the point 0.

It is the period-3 orbit $\{0, -1, 1\}$.

Exercise 3. Use Sharkovskii's Theorem to show that the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x-1)(1-3x^2/2)$ has a point of least period n for every $n \in \mathbb{N}$.

The map f is certainly continuous, so the existence (see Exercise 2) of an orbit of least period 3 implies, by Sharkovskii's Theorem, the existence of points of least period n for all $n \in \mathbb{N}$.

Exercise 4. Give an example of a continuous map $f : \mathbb{R} \to \mathbb{R}$ which has one fixed point, and no other periodic points.

We might choose $f(x) = 2x$ (or indeed $f(x) = cx$ for any $c \neq 1$). Alternatively, we might choose f to be any order reversing diffeomorphism.

Exercise 5. Give an example of a continuous map $f : \mathbb{R} \to \mathbb{R}$ which has three fixed points, and no other periodic points.

One such example is $f(x) = x^3$. Note that the three fixed points are at -1, 0, and 1. To see that there are no other periodic points, note that if $x \in (0,1)$ or $x \in (-1,0)$ then $f^n(x)$ converges to 0, if $x \in (1,\infty)$ then $f^n(x)$ converges to ∞, while if $x \in (-\infty,-1)$ then $f^n(x)$ converges to $-\infty$; therefore if $x \in \mathbb{R} \setminus \{-1,0,1\}$ then x is not periodic.

Exercise 6. Give an example of a continuous map $f : \mathbb{R} \to \mathbb{R}$ which has one fixed point, one orbit of least period two, and no other periodic points.

One such example is $f(x) = -x^3$. Note that the unique fixed point is at 0 (i.e. the unique real solution of $x^3 + x = 0$). The points -1 and 1 are of least period 2 (together with the point 0 they are the only real solutions of $0 = x^9 - x = x(x+1)(x-1)(x^2+1)(x^4+1)$). To see that there are no other periodic points, note that if $x \in (0,1)$ or $x \in (-1,0)$ then $f^n(x)$ converges to 0, while if $x \in (1,\infty)$ or $x \in (-\infty,-1)$ then $|f^n(x)|$ converges to ∞; therefore if $x \in \mathbb{R} \setminus \{-1,0,1\}$ then x is not periodic.

Exercise 7. Give an example of a map $f : \mathbb{R} \to \mathbb{R}$ which has one orbit of least period three, and no other periodic points.

Note that the question does not ask for f to be a *continuous* map (in fact a continuous map with the required properties does not exist, by Sharkovskii's Theorem). However we can easily exhibit a discontinuous map with these properties: for example let f be the map defined by setting $f(-1) = 0$, $f(0) = 1$, $f(1) = -1$, and $f(x) = 0$ for all $x \in \mathbb{R} \setminus \{-1, 0, 1\}$. Then $\{-1, 0, 1\}$ is an orbit of least period 3, and all points in $\mathbb{R} \setminus \{-1, 0, 1\}$. are preperiodic but not periodic.

Exercise 8. For the following values of μ, describe the behaviour of the orbit of the point x_0 under the logistic map $f_\mu(x) = \mu x(1 - x)$.

(a) $\mu = 4/5 = 0.8$, $x_0 = 3/5 = 0.6$,

(b) $\mu = 7/5 = 1.4$, $x_0 = 1/2 = 0.5$,

(c) $\mu = 33/10 = 3.3$, $x_0 = 13/20 = 0.65$,

(d) $\mu = 4$, $x_0 = 13/20 = 0.65$,

(e) $\mu = 4$, $x_0 = 33/50 = 0.66$.

(a) Letting $x_i = f_{0.8}^i(x_0)$ then $x_0 = 0.6$, $x_1 = 0.192$, $x_2 = 0.124109$, $x_3 = 0.869646$, $x_4 = 0.0635214$, etc., and $x_i \to 0$ as $i \to \infty$.

(b) Letting $x_i = f_{1.4}^i(x_0)$ then $x_0 = 0.5$, $x_1 = 0.35$, $x_2 = 0.3185$, $x_3 = 0.303881$, $x_4 = 0.296152$, $x_5 = 0.291824$, $x_6 = 0.289328$, etc., and $x_i \to (\mu - 1)/\mu = 2/7 = 0.285714\ldots$ as $i \to \infty$.

(c) Letting $x_i = f_{3.3}^i(x_0)$ then $x_0 = 0.65$, $x_1 = 0.75075$, $x_2 = 0.617511$, $x_3 = 0.779431$, $x_4 = 0.56733$, $x_5 = 0.81004$, $x_6 = 0.507788$, $x_7 = 0.8248$, $x_8 = 0.476867$, etc., and the orbit is in the basin of attraction of the attracting 2-cycle $\{0.479427, 0.823603\}$.

(d) Letting $x_i = f_4^i(x_0)$ then $x_0 = 0.65$, $x_1 = 0.91$, $x_2 = 0.3276$, $x_3 = 0.881113$, $x_4 = 0.419012$, $x_5 = 0.973764$, $x_6 = 0.102192$, and x_i does not seem to settle to any discernible pattern.

(e) Letting $x_i = f_4^i(x_0)$ then $x_0 = 0.66$, $x_1 = 0.8976$, $x_2 = 0.367657$, $x_3 = 0.929941$, $x_4 = 0.260602$, $x_5 = 0.770754$, $x_6 = 0.706768$, and x_i does not seem to settle to any discernible pattern. Moreover note that this orbit of $x_0 = 0.66$ is already looking rather different to the orbit of 0.65 in (d) above; this is a hallmark of *chaos*, that two orbits which start close to each other end up looking nothing like each other.

Exercise 9. Let $f(x) = 1 - (13/10)x^2 = 1 - 1.3x^2$.

Use a computer to determine numerically the first 50 points in the f-orbit of the point 0, and the first 50 points in the f-orbit of the point $1/3$.

What is the period of the attracting orbit of f?

The points $f^n(0)$ for $0 \leq n \leq 50$, listed in order, and to six-digit precision, are: 0, 1, -0.3, 0.883, -0.0135957, 0.99976, -0.299375, 0.883487, -0.0147135, 0.999719, -0.299268, 0.88357, -0.0149047, 0.999711, -0.299249, 0.883585, -0.0149389, 0.99971, -0.299246, 0.883588, -0.014945, 0.99971, -0.299245, 0.883588, -0.0149461, 0.99971, -0.299245, 0.883588, -0.0149463, 0.99971, -0.299245, 0.883588, -0.0149464, 0.99971, -0.299245, 0.883588, -0.0149464, 0.99971, -0.299245, 0.883588, -0.0149464, 0.99971,

-0.299245, 0.883588, -0.0149464, 0.99971, -0.299245, 0.883588, -0.0149464, 0.99971, -0.299245.

The points $f^n(1/3)$ for $0 \leq n \leq 50$, listed in order, and to six-digit precision, are: 0.333333, 0.855556, 0.0484321, 0.996951, -0.292084, 0.889093, -0.0276328, 0.999007, -0.29742, 0.885003, -0.0182004, 0.999569, -0.298881, 0.883872, -0.0155975, 0.999684, -0.299178, 0.88364, -0.0150665, 0.999705, -0.299233, 0.883598, -0.0149681, 0.999709, -0.299243, 0.88359, -0.0149503, 0.999709, -0.299245, 0.883588, -0.0149471, 0.99971, -0.299245, 0.883588, -0.0149465, 0.99971, -0.299245, 0.883588, -0.0149464, 0.99971, -0.299245, 0.883588, -0.0149464, 0.99971, -0.299245, 0.883588, -0.0149464, 0.99971, -0.299245, 0.883588, -0.0149464.

We observe that there is an attracting period-4 orbit. (Note that the orbits of the points 0 and $1/3$ get very close to, but do not ever reach, the attracting period-4 orbit: this can be seen e.g. by using higher precision calculations).

Exercise 10. Let $f(x) = 1 - \bar{\lambda}x^2$ where the value

$$\bar{\lambda} = \frac{1}{3}\left(2 + \left(25/2 - 3\sqrt{69}/2\right)^{1/3} + \left(25/2 + 3\sqrt{69}/2\right)^{1/3}\right) \approx 1.75487766624669276$$

is the only real root of the polynomial $1 - \lambda + 2\lambda^2 - \lambda^3$. (The value $\bar{\lambda}$ is chosen so that 0 is a period-3 point).

Use a computer to determine numerically the first 100 points in the f-orbit $\{f^n(1/3)\}_{n=0}^{\infty}$ of the point $1/3$.

What is the smallest value $n \in \mathbb{N}$ such that $|f^n(1/3)| < 1/100$?

What is the smallest value $n \in \mathbb{N}$ such that $|f^n(1/3)| < 1/1000$?

What is the smallest value $n \in \mathbb{N}$ such that $|f^n(1/3)| < 10^{-6}$?

The points $f^n(1/3)$ for $1 \leq n \leq 100$, listed in order, and to 30-digit precision, are:
0.805013592639256359994499011516, -0.13724300399842469417029663821,
0.96694575226767184768572001197, -0.64078269402626004164645869879,
0.27944291154913380895156940089, 0.86296451470939636812027419337,
-0.3068710047569593686982052890, 0.8347434973479832615779149072,
-0.2227929779137414332615638862, 0.9128936444266934943780011606,
-0.462470834722877354274795755, 0.624668040593139431299310199,
0.315228903426547703378679718, 0.825619102184820392690376406,
-0.19620692439734415711846210, 0.93244219864825837155853729,
-0.52577567356143373447667293, 0.51488151456471208844976748,
0.5347767116008343248170330, 0.4981293253769257634425915,
0.5645572474949496531308910, 0.440676816409665405875317,
0.659209677538190472251826, 0.237404925862833321711402,
0.901093166428749540446619, -0.42490607879351239322009,
0.68316530324758072393065, 0.18097256558688413584035,
0.94252587959735727599594, -0.55895440835686168480563,
0.4517235169974377326302, 0.6419099543643334229536,
0.2769054838324370250360, 0.8654418646983001916738,
-0.314385158446140295665, 0.826551352348052206649,
-0.198909850463220240671, 0.930568039437850485450,
-0.51964838166587839212, 0.52612259112985583610,
0.51424112111767277215, 0.53593325213070198278,
0.4959562261974700898,

0.5683482158251312064, 0.4331399524841365631,
0.6707670177035603671, 0.210430713400598154,
0.922292112646636361, -0.492738850670532420,
0.573930487549630478, 0.42194995730815767,
0.68755848035855309, 0.17040498648317265,
0.94904210903116106, -0.58058483919556888,
0.4084680202002492, 0.7072054641238396,
0.1223162412565804, 0.9737448139215746,
-0.663938085140423, 0.226425740924948,
0.910029850533003, -0.453309535923899,
0.639391003750132, 0.28256941087727,
0.85988096450028, -0.29754825130899,
0.84463197275508, -0.2519352890099,
0.8886154755092, -0.3857171488023,
0.7389133039021, 0.041849305308,
0.9969265698084, -0.744107254770,
0.028331946076, 0.9985913611565,
-0.7499371706938, 0.013046692428,
0.9997012914215, -0.7538294288025,
0.002775309669, 0.99998648332896,
-0.75483022635905, 0.00012568499267,
0.99999997227869346, -0.7548775689516907,
$2.577768562 \times 10^{-7}, 0.9999999999998833902961$,
-0.754877666246283488519, $1.084338691 \times 10^{-12}$,
0.999999999999999999999997936632, -0.75487766624669276004950165444,
1.918700×10^{-23}, 1.00000000000000000000000000000,
-0.75487766624669276004950889636, 0×10^{-30},
1.00000000000000000000000000000, -0.75487766624669276004950889636, 0×10^{-30}

Although the orbit containing 0 is attracting, we see that the orbit of the point $1/3$ takes some time to become close to this orbit. We see that the smallest value n such that $|f^n(1/3)| < 1/100$ is $n = 82$ (note that $f^{82}(1/3) \approx 0.002775309669$), the smallest value n such that $|f^n(1/3)| < 1/1000$ is $n = 85$ (note that $f^{85}(1/3) \approx 0.00012568499267$), and the smallest value n such that $|f^n(1/3)| < 10^{-6}$ is $n = 88$ (note that $f^{88}(1/3) \approx 2.577768562 \times 10^{-7}$).

Note that once $n > 80$, the convergence of $f^n(1/3)$ to the period-3 orbit is very rapid; in particular, the computed numerical values of $f^{97}(1/3)$ and $f^{100}(1/3)$ are indistinguishable from 0 (to 30 digit precision), and $f^{95}(1/3)$ and $f^{98}(1/3)$ are indistinguishable from 1 (to 30 digit precision).