Defn A function $f : \mathbb{R} \to \mathbb{R}$ is said to be C^1 if f is differentiable, and its derivative f' is continuous.

Theorem If $f : \mathbb{R} \to \mathbb{R}$ is C^1, and p is a fixed point of f, then

- p is attracting if $|f'(p)| < 1$
- p is repelling if $|f'(p)| > 1$.

Note This theorem is inspired by the example of $f(x) = ax$, where we saw that 0 is attracting if $|a| < 1$ and 0 is repelling if $|a| > 1$.

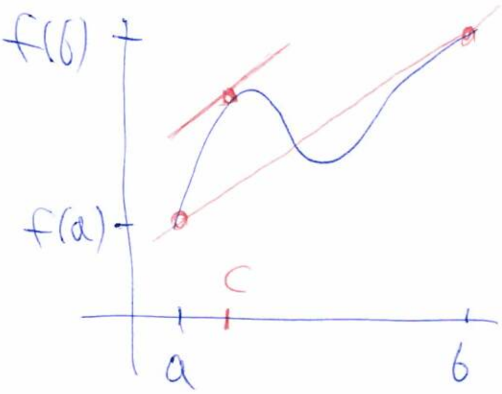
(In this case $f'(x) = a$, so in particular $f'(0) = a$)

A useful tool in proving the Theorem is :

Lemma (Mean Value Theorem)

If $f : [a,b] \to \mathbb{R}$ is C^1 (on (a,b)) then exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$



Proof of Theorem We shall prove the attracting case (leaving the repelling case as an exercise).

Suppose $|f'(p)| < 1$.

We must show that p is attracting,

i.e. that there exists $\delta > 0$ such that

if $x \in (p-\delta, p+\delta) =: I$

then $\lim_{n\to\infty} f^n(x) = p$.

Since $|f'(p)| < 1$, let us choose

some $K \in (|f'(p)|, 1)$.

Now since f' is continuous (i.e. f is C^1)

we can choose $\delta > 0$ such that if

$I = (p-\delta, p+\delta)$ then

$$|f'(x)| < K \quad \text{for all } x \in I \qquad (*)$$

Now by the Mean Value Theorem,

for all $x \in \mathbb{R} \setminus \{p\}$,

$$f(x) - f(p) = f'(c)(x-p) \qquad (**)$$

for some $c \in \begin{cases} (p,x) & \text{if } x > p \\ (x,p) & \text{if } p > x \end{cases}$

Therefore, for all $x_0 \in I \setminus \{p\}$,

$$|\underbrace{f(x_0)}_{x_1} - \underbrace{f(p)}_{p}| \overset{\text{by } (**)}{=} |f'(c)|\,|x_0 - p|$$

$$\overset{\text{by } (*)}{<} K\,|x_0 - p|$$

i.e. $|x_1 - p| < K\,|x_0 - p| \quad (***)$

In particular,

$$|x_1 - p| \overset{\text{since } K<1}{<} |x_0 - p| \overset{\text{since } x_0 \in I = (p-\delta,\, p+\delta)}{<} \delta$$

therefore $x_1 \in I = (p-\delta,\, p+\delta)$.

Applying the Mean Value Theorem again (on $[p, x_1]$ or $[x_1, p]$) gives:

$$|f(x_1) - f(p)| < K\,|x_1 - p|$$

i.e. $|x_2 - p| < K\,|x_1 - p|$

$$\overset{\text{by } (***)}{<} K^2\,|x_0 - p|$$

Repeating this argument inductively we deduce

$$|x_n - p| < K^n |x_0 - p| \quad \forall n \in \mathbb{N}$$

Taking limits as $n \to \infty$, and recalling that $K \in (0,1)$ (so that $\lim_{n\to\infty} K^n = 0$) we get that

$$\lim_{n\to\infty} |x_n - p| = 0$$

i.e. $$\lim_{n\to\infty} x_n = p$$

i.e. $$\lim_{n\to\infty} f^n(x_0) = p$$

So p is attracting. $\square$

We can generalise the notions of 'attracting' and 'repelling' to periodic points / periodic orbits.

<u>Defn</u> Suppose $p \in \mathbb{R}$ is a periodic point of least period K for the function $f: \mathbb{R} \to \mathbb{R}$ (so $f^k(p) = p$).

We say p is <u>attracting</u> (for f) if p is an attracting fixed point for f^k.

We say p is <u>repelling</u> (for f) if p is a repelling fixed point for f^k

It can be shown that if a point p (of least period k under f) is attracting then so are the other points in its orbit

i.e. so are $f(p), f^2(p), \dots, f^{k-1}(p)$

So we call ~~this~~ the orbit (i.e. $\{x, f(x), \dots, f^{k-1}(x)\}$) an <u>attracting periodic orbit</u>, or an <u>attracting cycle</u>.

Similarly, if p is repelling then so are $f(p), \dots, f^{k-1}(p)$, so we call the orbit $\{p, f(p), \dots, f^{k-1}(p)\}$ a <u>repelling periodic orbit</u>, or a <u>repelling cycle</u>.

A consequence of the previous theorem is:

<u>Corollary / Theorem</u> If $f: \mathbb{R} \to \mathbb{R}$ is C^1, and p is a periodic point of least period k, then:

- p is attracting if $|(f^k)'(p)| < 1$ (i.e. $\{p, f(p), \dots, f^{k-1}(p)\}$ is attracting if $|(f^k)'(p)| < 1$)
- and p is repelling if $|(f^k)'(p)| > 1$ (i.e. $\{p, f(p), \dots, f^{k-1}(p)\}$ is a repelling periodic orbit if $|(f^k)'(p)| > 1$)

Definition If x_0 is a periodic point of least period k (i.e. $x_0, x_1, \ldots, x_{k-1}$ is a k-cycle) then we refer to the value $(f^k)'(x_0)$ as its multiplier.

We can show that

$$(f^k)'(x_0) = f'(x_0).f'(x_1)\ldots f'(x_{k-1})$$
$$= \prod_{i=0}^{k-1} f'(x_i) \qquad (*)$$

i.e. The multiplier is equal to the product of the derivatives (of f) at all points in the orbit.

In particular: The multiplier is the same value for all points in the orbit

So it makes sense to talk of the multiplier associated to a periodic <u>orbit</u>.

To justify (*), first note that

$$(f^2)'(x) = (f \circ f)'(x)$$

$$\overset{\text{Chain Rule}}{=} f'(f(x)) \cdot f'(x)$$

So $(f^2)'(x_0) = f'(f(x_0)) \cdot f'(x_0)$

$$= f'(x_1) \cdot f'(x_0)$$

(which is (*) for $k=2$)

Similarly for $k=3$:

$$(f^3)'(x) = (f \circ f \circ f)'(x)$$

Chain Rule

$$= f'(f \circ f(x)) \cdot (f \circ f)'(x)$$

by the above

$$= f'(f^2(x)) \cdot f'(f(x)) \cdot f'(x)$$

So $(f^3)'(x_0) = f'(f^2(x_0)) \cdot f'(f(x_0)) \cdot f'(x_0)$

$$= f'(x_2)\, f'(x_1)\, f'(x_0)$$

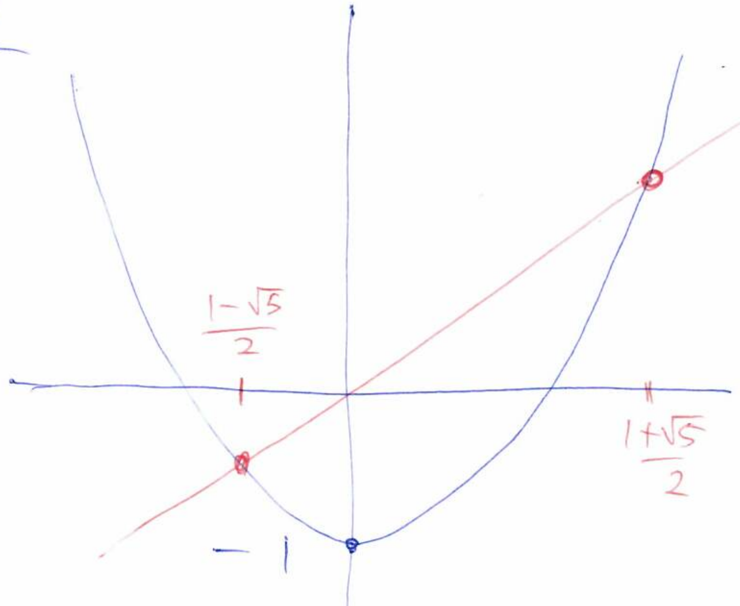
(which is $(*)$ for $k = 3$)

To prove $(*)$ in general, we can use induction (Exercise).

<u>Example</u> $f(x) = x^2 - 1$

Recall the fixed points are at

$$x = \frac{1 \pm \sqrt{5}}{2}$$



Note that $f'(x) = 2x$

So $f'\left(\frac{1+\sqrt{5}}{2}\right) = 1+\sqrt{5} > 1$,

thus the fixed point $\frac{1+\sqrt{5}}{2}$ is repelling (by the Theorem).

For the other fixed point $\frac{1-\sqrt{5}}{2}$, note that $f'\left(\frac{1-\sqrt{5}}{2}\right) = 1-\sqrt{5}$, which is larger than 1 in absolute value, so this fixed point is repelling as well, again by the Theorem.

We also saw that $\{-1, 0\}$ is a 2-cycle.

We can use the Corollary / Theorem to determine whether this 2-cycle is attracting or repelling.

Now $f'(-1) = 2(-1) = -2$

but $f'(0) = 2.0 = 0$

So the multiplier for this 2-cycle

is $(f^2)'(0) = f'(0)\, f'(-1)$

$= 0.(-2)$

$= 0$

< 1

Therefore, by the Corollary / Theorem,

this 2-cycle is <u>attracting</u>.

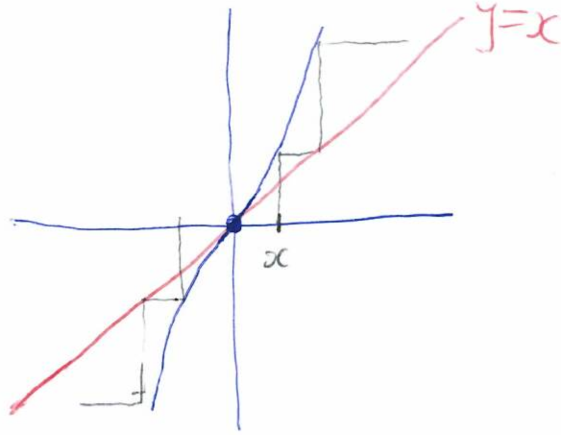
Therefore $|(f^2)'(0)| = |0| < 1$

__Remark__ If a fixed point p satisfies $|f'(p)| = 1$, or more generally if p is a point of least period k and $|(f^k)'(p)| = 1$, then we cannot immediately tell whether it is attracting, or repelling, or neither.

We __could__ examine higher order derivatives to determine the behaviour, or we could analyse behaviour graphically.

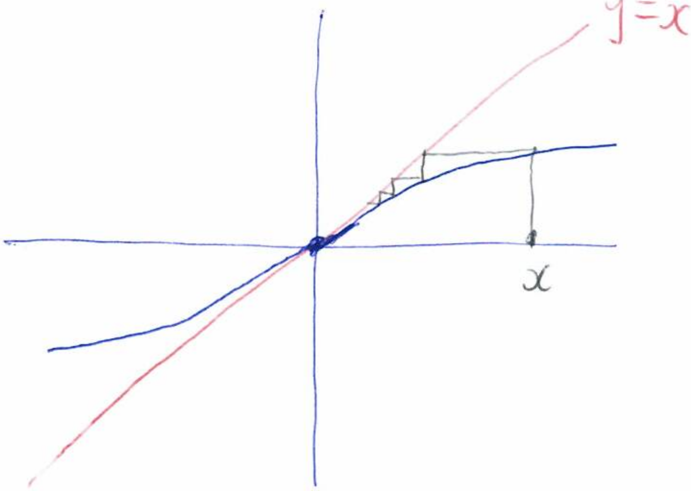
__Examples__ For each of the following functions f, the point 0 is a fixed point, and $f'(0) = 1$.

1. $f(x) = x + x^3$



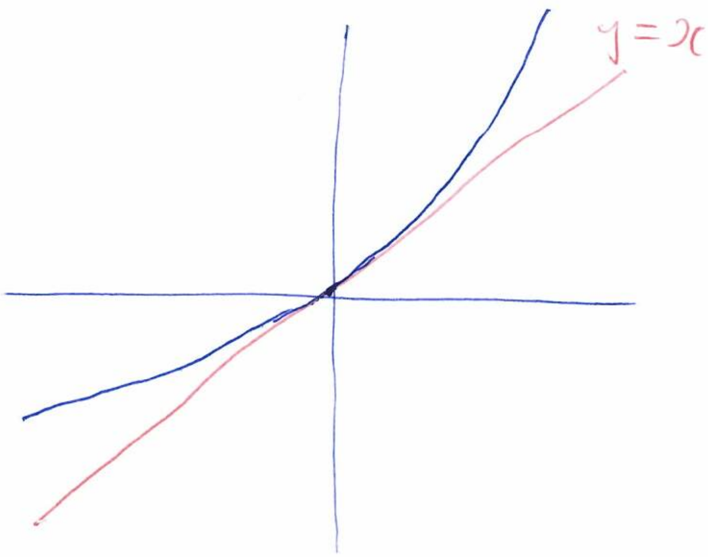
Here 0 is repelling

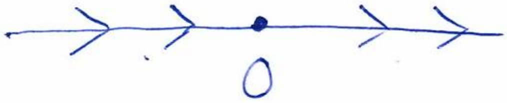
2. $f(x) = x - x^5$



Here 0 is attracting

3. $f(x) = x + x^2$

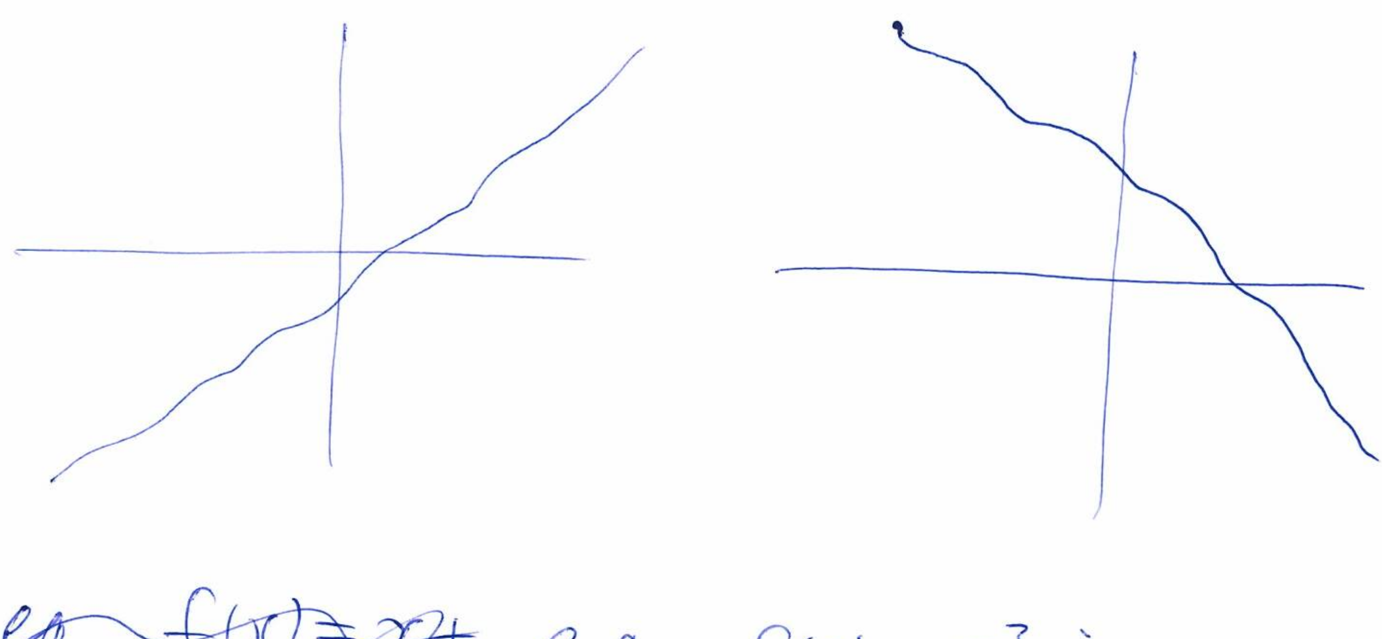




Here 0 is neither attracting nor repelling

Diffeomorphisms

<u>Defn</u> A <u>diffeomorphism</u> (or C^1-diffeomorphism) is a bijective function $f: \mathbb{R} \to \mathbb{R}$ for which both f and f^{-1} are C^1 (i.e. both f and f^{-1} are differentiable with continuous derivative)



~~e.g. $f(x) = x+$~~ e.g. $f(x) = x^3$ is <u>not</u> a diffeomorphism, since its inverse function $f^{-1}(x) = x^{1/3}$ is not differentiable at 0 ("derivative $= +\infty$")

Note that for a diffeomorphism f,

$$f^{-1}(f(x)) = x$$

Differentiating both sides gives :

$$(f^{-1})'(f(x)) \cdot f'(x) = 1$$

So $(f^{-1})'(f(x)) = \dfrac{1}{f'(x)}$

we see that neither f' nor $(f^{-1})'$ can be equal to zero at any point

So by the Intermediate Value Theorem applied to the continuous function f', we see that

either $f'(x) > 0$ for all $x \in \mathbb{R}$

or $f'(x) < 0$ for all $x \in \mathbb{R}$.

__Defn__ We say that $f: \mathbb{R} \to \mathbb{R}$ is __order-preserving__ if whenever $a < b$ then $f(a) < f(b)$, and we say it is __order-reversing__ if whenever $a < b$ then $f(a) > f(b)$.

__Lemma__ If $f: \mathbb{R} \to \mathbb{R}$ is a diffeomorphism then it is either order-preserving or order-reversing.

If $f' > 0$ then f is order-preserving.

If $f' < 0$ then f is order-reversing.

Proof Suppose $a, b \in \mathbb{R}$ with $a < b$.

If $f'(x) > 0$ for all $x \in \mathbb{R}$ then

$$\frac{f(b) - f(a)}{b - a} = \underline{f'(c)}$$

for some $c \in (a, b)$ by the Mean Value Theorem.

So $$\frac{f(b) - f(a)}{b - a} > 0 \quad (\text{since } f'(c) > 0)$$

So $f(b) - f(a) > 0$ using that $b - a > 0$

i.e. $f(b) > f(a)$

So f is order-preserving.

A similar argument shows that if $f' < 0$ then f is order-reversing. $\square$

Remark Later in this module we will see that many functions have a great variety of periodic orbit behaviour, involving the presence of lots of different periods (i.e. lots of periodic points, with their least periods being wide-ranging).

By contrast, we will see that diffeomorphisms do not have this property, and in fact diffeomorphisms will not have any periodic orbits of least period > 2.

Think of diffeomorphisms as 'non-chaotic'.