Fixed points: We need to solve the equation $f(x) = x$, i.e. $-x^3 = x$

i.e. $x^3 + x = 0$

i.e. $x(x^2+1) = 0$

The only (real) solution is $x = 0$

i.e. 0 is the only fixed point

What about other periodic points?

Let's examine the orbit of any point x_0:

$$x_0 \xrightarrow{f} -x_0^3 \xrightarrow{f} x_0^9 \xrightarrow{f} -x_0^{27} \rightarrow \dots$$

Notice that since the power goes up, the 'size' of x_n is either growing or shrinking unless $x_0 = 1$ or $x_0 = -1$

So the only candidates for periodic points (apart from the fixed point at 0) are $x_0 = 1$ and $x_0 = -1$

Let $x_0 = 1$. Then the orbit of x_0 under f is

$$1 \xrightarrow{f} -1 \xrightarrow{f} 1 \xrightarrow{f} -1 \xrightarrow{f} 1 \to \cdots$$

i.e.

$$\begin{tikzcd} 1 \arrow[r, bend left, "f"] & -1 \arrow[l, bend left, "f"] \end{tikzcd}$$

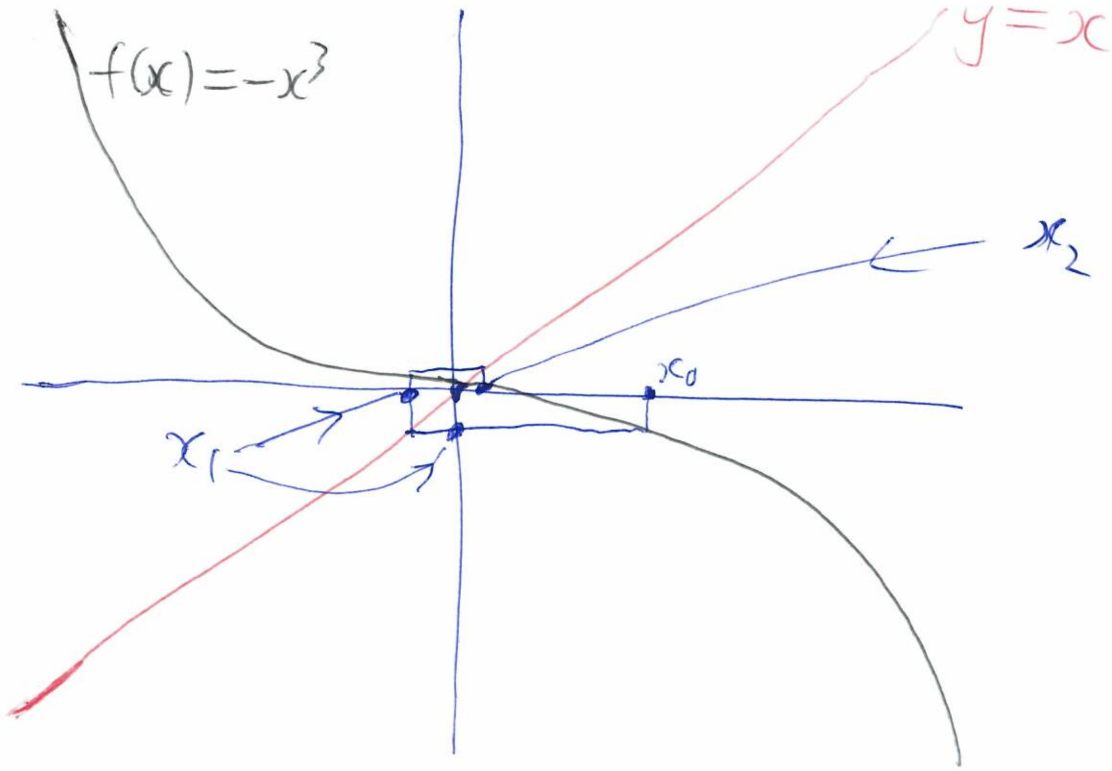
Thus 1 is a point of period 2 (i.e. period-2 point), and similarly -1 is a period-2 point.
We say $\{-1, 1\}$ is a period-2 orbit, or 2-cycle

What about other $x_0 \in \mathbb{R}$?

We notice that $|f^n(x_0)| \to 0$ as $n \to \infty$ if $|x_0| < 1$,

and $|f^n(x_0)| \to \infty$ as $n \to \infty$ if $|x_0| > 1$.

Thus 0 is an "attracting" fixed point



Definition Given a periodic point x_0, let m be the smallest natural number (i.e. smallest strictly positive integer) such that $f^m(x_0) = x_0$. Then m is called the least period (or prime period) of x_0

Example If $f(x) = -x^3$, then 0 is a fixed point, but it is also a period-2 point, and also a period-3 point, ... In fact a fixed point is a period-n point for any $n \geq 1$. The least period of this/any fixed point is 1.

For this f, we saw -1 is a period-2 point, but it is also a period-4 point, a period-6 point, etc. Its <u>least period</u> is 2.

<u>Definition</u> If x_0 has least period m, then the orbit $\{x_0, f(x_0), \dots, f^{m-1}(x_0)\}$ is often called an <u>m-cycle</u> (or a <u>cycle</u>). [or a <u>periodic orbit</u>]

<u>Question</u> If x_0 has least period 2, can it also have period 5?

<u>Answer</u> No!

Let $f : \mathbb{R} \to \mathbb{R}$.

<u>Lemma</u> Let $x \in Per_n(f)$

$$= \{x_0 : f^n(x_0) = x_0\}$$

and suppose x has least period k.

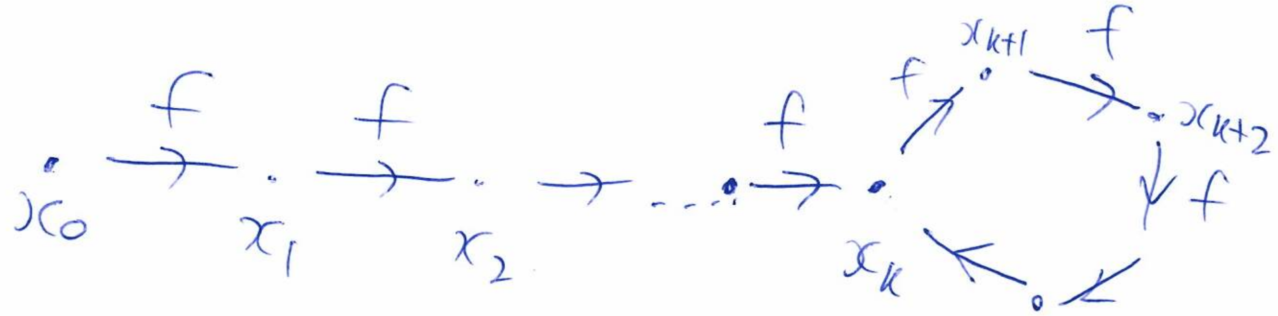
Then n must be an integer multiple of k. (i.e. k must be a factor of n)

<u>Proof</u> Omitted / exercise. $\square$

<u>Defn</u> We say that a point x_0 is <u>eventually periodic</u> (or <u>pre-periodic</u>) (of period n) if the exists some integer $k \geq 0$ such that $f^k(x_0)$ is periodic (of period n).

i.e. $f^n(f^k(x_0)) = f^k(x_0)$

i.e. $x_{k+n} = x_k$



<u>Example</u> For $f(x) = x^2$, the point -1 is eventually periodic (of period 1) [we could say it is an <u>eventually fixed point</u>] since $f(-1) = 1$, and 1 is a fixed point.

<u>Example</u> Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 - 2$.

Does f have any fixed points? If so, what are they?

Fixed points are solutions of the equation

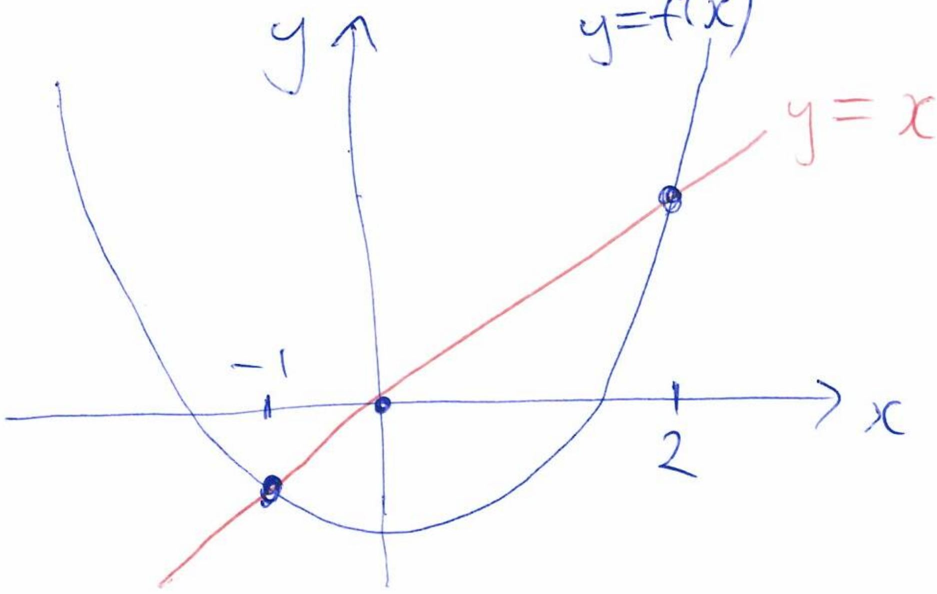
$$f(x) = x$$

i.e. $x^2 - 2 = x$

i.e. $x^2 - x - 2 = 0$

i.e. $(x-2)(x+1) = 0$

i.e. $x = 2$ and $x = -1$ are the <u>only</u> fixed points of f



Note that -2 is an eventually fixed point since $f(-2) = (-2)^2 - 2$
$= 2$

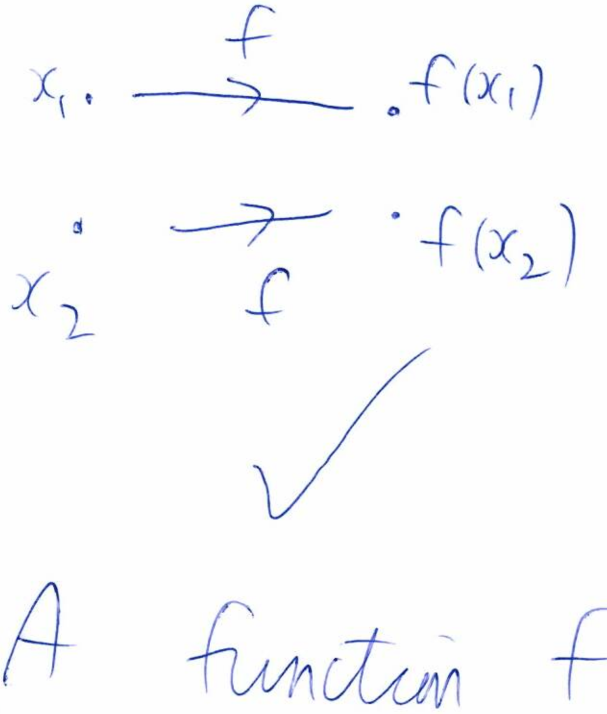
Note also that 0 is an eventually fixed point, since $f(0) = 0^2 - 2$
$= -2$,
so $f(f(0)) = f(-2) = 2$

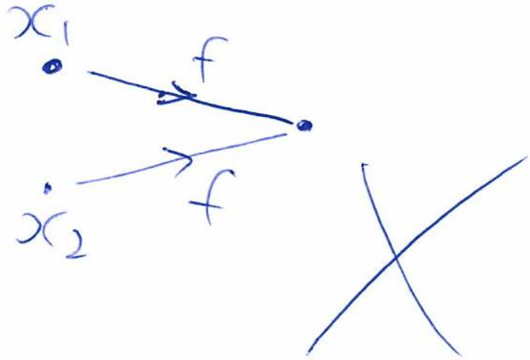
We can represent this as

$$0 \xrightarrow{f} -2 \xrightarrow{f} 2 \circlearrowleft f$$

<u>Recall</u> Let A, B be sets $\left(\begin{array}{c}\text{e.g.}\\ A=B=\mathbb{R}\end{array}\right)$

A function $f: A \to B$ is said to be <u>one-to-one</u> (or <u>injective</u>) if, for all $x_1, x_2 \in A$ with $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$





A function $f: A \to B$ is <u>onto</u> (or <u>surjective</u>) if for all $y \in B$ there exists $x \in A$ such that $f(x) = y$.

A function $f : A \to B$ is <u>bijective</u> (or <u>invertible</u>) if it is both injective and surjective.

<u>Lemma</u> If $f : \mathbb{R} \to \mathbb{R}$ is one-to-one then every eventually periodic point is actually a periodic point.

<u>Remark</u> Among our examples so far, $f(x) = x^2$ and $f(x) = x^2 - 2$ are <u>not</u> one-to-one, but $f(x) = -x^3$ <u>is</u> one-to-one.

<u>Corollary</u>: If $f : \mathbb{R} \to \mathbb{R}$ is invertible (i.e. bijective) then every eventually periodic point is actually a periodic point.

Definition Let p be a fixed point of f. The **basin of attraction** of p is defined to be the set of points

$$\text{Basin}(p) := \{ x_0 \in \mathbb{R} : \lim_{n\to\infty} f^n(x_0) = p \}$$

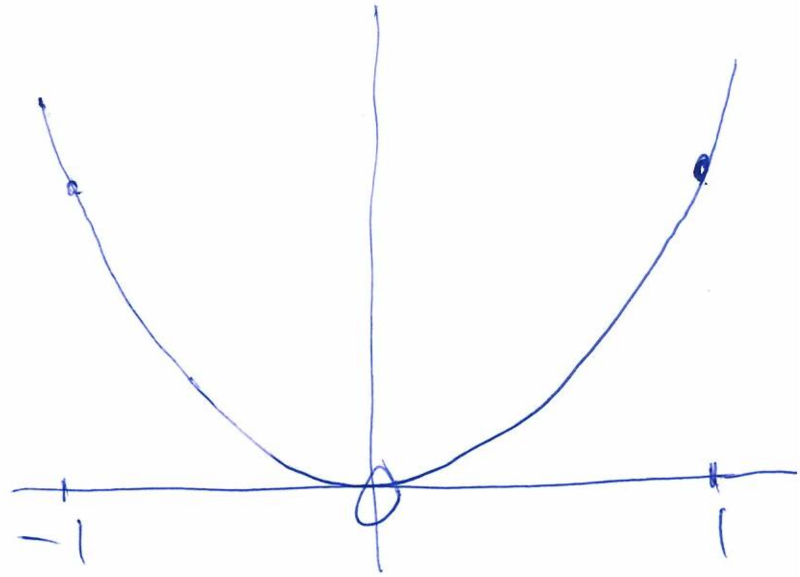
Example $f(x) = x^2$. Recall this f has fixed points at 0 and 1.

There

$\text{Basin}(0)$

$= (-1, 1)$

$= \{ x_0 : -1 < x_0 < 1 \}$



and $\text{Basin}(1) = \{-1, 1\}$

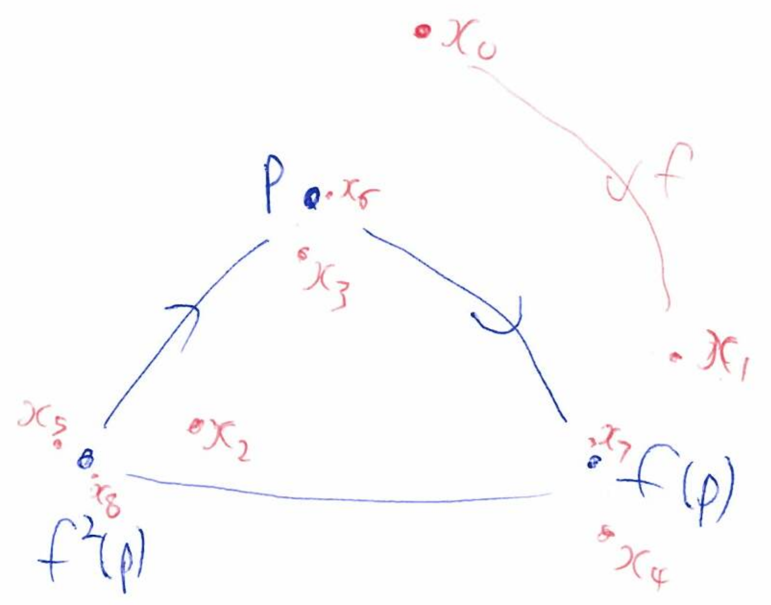
We can extend the definition of basin of attraction to periodic orbits:

Definition Let p be a periodic point of least period k under the map f. The basin of attraction of the k-cycle $\{p, f(p), \dots, f^{k-1}(p)\}$ is the set

$$\text{Basin}(\{p, f(p), \dots, f^{k-1}(p)\})$$

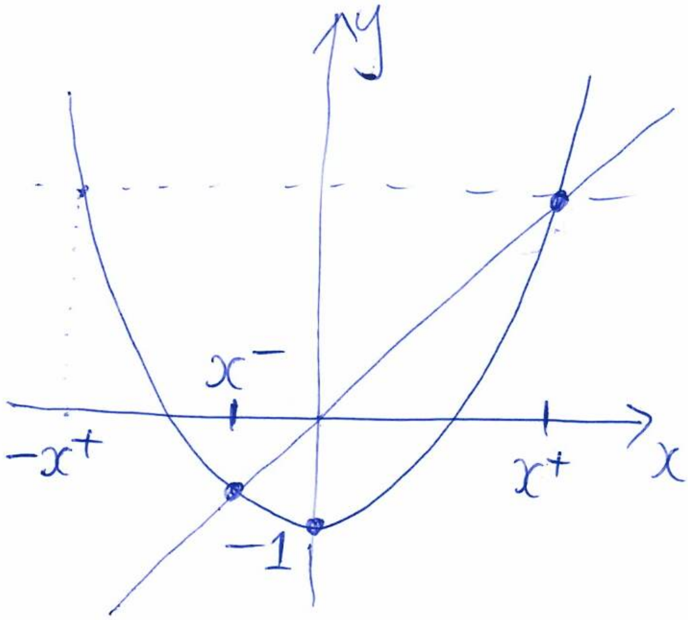
$$:= \bigcup_{i=0}^{k-1} \{x_0 \in \mathbb{R} : \lim_{n\to\infty} f^{nk}(x_0) = f^i(p)\}$$

Schematic diagram (illustrating a point x_0 in the basin of attraction of a periodic orbit)



Example Consider the map $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 - 1$.

Fixed points:



$$f(x) = x$$

i.e. $x^2 - 1 = x$

i.e. $x^2 - x - 1 = 0$

i.e. $x = \frac{1}{2}\left(1 \pm \sqrt{1+4}\right)$

$= \frac{1}{2}(1 \pm \sqrt{5})$

Let us write $x^+ = \frac{1}{2}(1+\sqrt{5}) \approx 1.62$

$x^- = \frac{1}{2}(1-\sqrt{5}) \approx -0.62$

These are the 2 fixed points of f.

Claim: Points in the union

$$(-\infty, -x^+) \cup (x^+, \infty)$$

have orbits which converge to ∞.

(i.e. if $x_0 \in (-\infty, -x^+) \cup (x^+, \infty)$ then $f^n(x_0) \longrightarrow \infty$ as $n \to \infty$)

Question: What happens if $x_0 \in (-x^+, x^+)$?

It turns out that there is a 2-cycle $\{0, -1\}$

The fixed point $x^+ = \frac{1+\sqrt{5}}{2}$ is "repelling", and the fixed point $x^- = \frac{1-\sqrt{5}}{2}$ is also "repelling".

Question: if x_0 is in $(-1,0)$ what happens to the orbit x_n ?

$$(x_n = f^n(x_0))$$

Let us experiment:

$$x_0 = -0.5$$

$$x_1 = (-0.5)^2 - 1 = -0.75$$

$$x_2 = (-0.75)^2 - 1 = -0.4375$$

$$x_3 = -0.8085\ldots$$

$$x_4 = -0.346\ldots$$

$$x_5 = -0.8801\ldots$$

$$x_6 = -0.225\ldots$$

$$x_7 = -0.949\ldots$$

$$x_8 = -0.098\ldots$$

$$x_9 = -0.9902\ldots$$

$$x_{10} = -0.0194\ldots$$

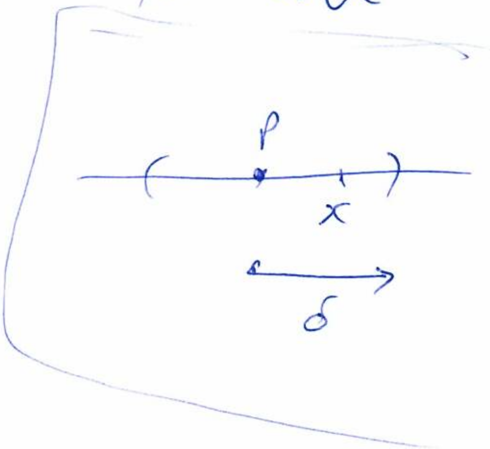
We 'see' that $x_0 = -\frac{1}{2}$ belongs to the basin of attraction of the 2-cycle $\{-1, 0\}$.

In fact, Basin$(\{-1, 0\})$ is equal to $(-x^+, x^+) \setminus \{x^-\}$

$$= (-x^+, x^-) \cup (x^-, x^+)$$

Attracting and repelling fixed points

<u>Defn</u> Let p be a fixed point of a function $f : \mathbb{R} \to \mathbb{R}$. We say that p is <u>attracting</u> if there exists some $\delta > 0$ such that for all points $x \in (p-\delta, p+\delta)$, we have that



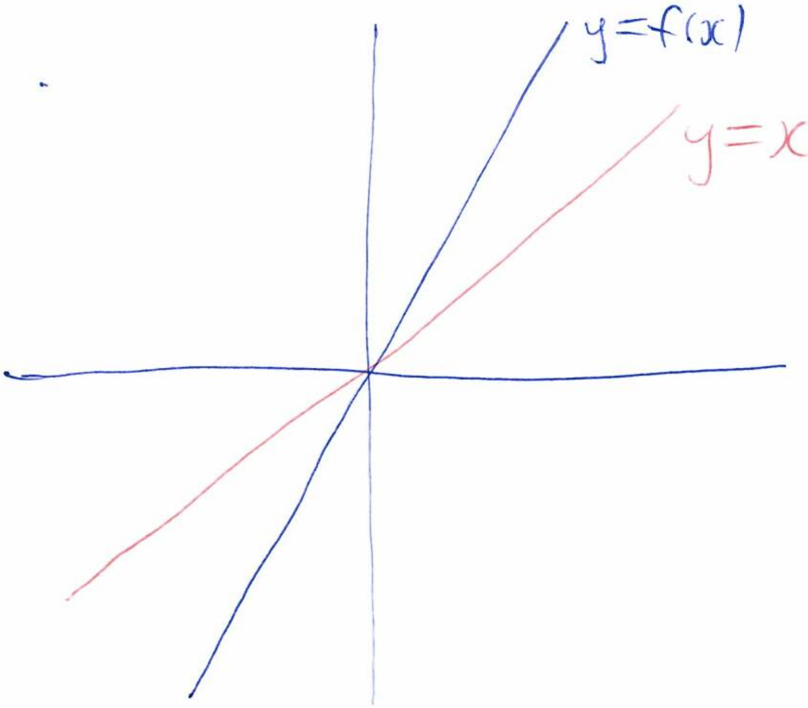
$$\lim_{n\to\infty} f^n(x) = p.$$

In other words, p is attracting if there exists $\delta > 0$ such that $(p-\delta, p+\delta)$ is contained in the basin of attraction of p.

<u>Defn</u> The fixed point p is <u>repelling</u> if there exists $\delta > 0$ such that if $I = (p-\delta, p+\delta)$ then for all $x \in I \setminus \{p\}$ there exists $N \in \mathbb{N}$ such that $f^N(x) \notin I$.

In other words, p is repelling if there exists an open interval I centred on p for which every point in $I \setminus \{p\}$ escapes from that interval (under iteration by f).

<u>Example</u> Let $f(x) = ax$, where $a \in \mathbb{R}$.



Note that 0 is a fixed point of f.

Note that $f^2(x) = f(f(x))$
$$= f(ax) = a^2 x$$

More generally, $f^n(x) = a^n x$

- Suppose $|a| < 1$. In this case 0 is an <u>attracting</u> fixed point
- Suppose $|a| > 1$. In this case 0 is <u>repelling</u>