Chapter 1 Dynamical systems (on $\mathbb{R}$)

Let $f : \mathbb{R} \to \mathbb{R}$

This defines a recurrence relation:

$$\boxed{x_{n+1} = f(x_n) \quad \text{for all } n \geqslant 0}$$

which is a dynamical rule for generating a sequence $(x_n)_{n=0}^{\infty}$ provided we have a starting point $x_0 \in \mathbb{R}$.

This is because

$$x_1 = f(x_0),$$

$$x_2 = f(x_1) = f(f(x_0))$$

$$= f^2(x_0), \quad \text{notation}$$

$$x_3 = f(x_2) = f(f(x_1))$$

$$= f(f(f(x_0)))$$

$$= f^3(x_0)$$

<u>Example</u> Take $f(x) = \cos(x)$

Choose $x_0 = 0.8$

$$x_1 = f(x_0) = \cos(0.8) = 0.697\ldots$$

$$x_2 = f^2(x_0) = f(0.697\ldots) = 0.7669\ldots$$

$$x_3 = f^3(x_0) = 0.7200\ldots$$

$$x_4 = f^4(x_0) = 0.7517\ldots$$

Continuing in this way, we observe that for large values of n,

$$x_n = f^n(x_0) \simeq 0.739085\ldots$$

You can think of x_n as representing "position at time n".

In particular, x_0 is our "initial position", i.e. 'position at time zero'.

<u>Note</u> The dynamical rule (namely $x_{n+1} = f(x_n) \ \forall n \geq 0$) is <u>deterministic</u>.

Defn A point $x_0 \in \mathbb{R}$ is called a fixed point of f if

$$f(x_0) = x_0 .$$

For a given function f, define

$$\mathrm{Fix}(f) := \{ x_0 \in \mathbb{R} : f(x_0) = x_0 \}$$

to be the set of all fixed points of f.

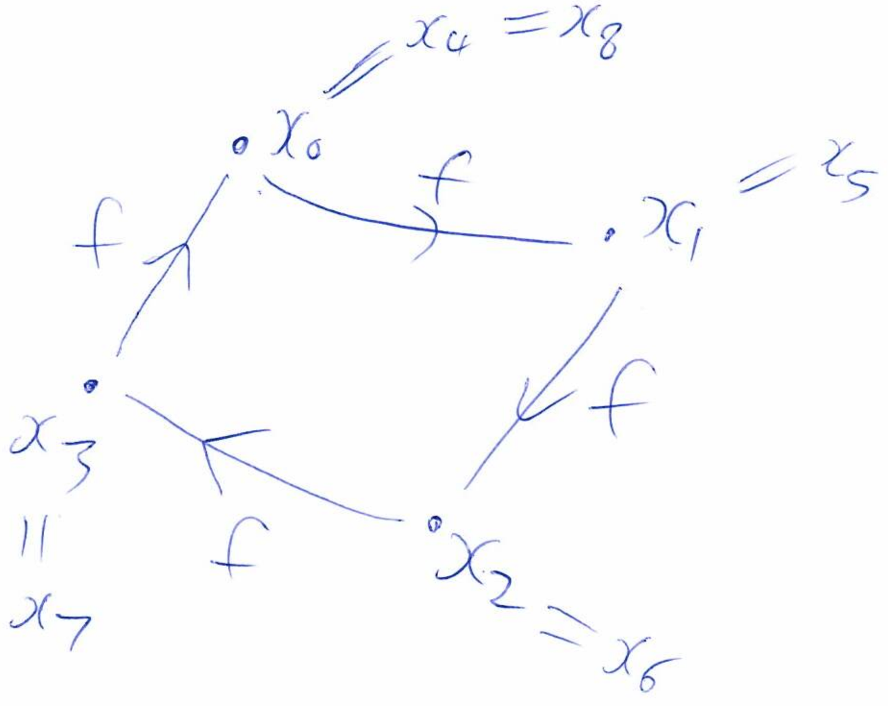
A point $x_0 \in \mathbb{R}$ is said to be periodic (with period n) if $f^n(x_0) = x_0$.

i.e. $\underbrace{(f \circ f \circ \ldots \circ f)}_{n \text{ times}}(x_0) = x_0$

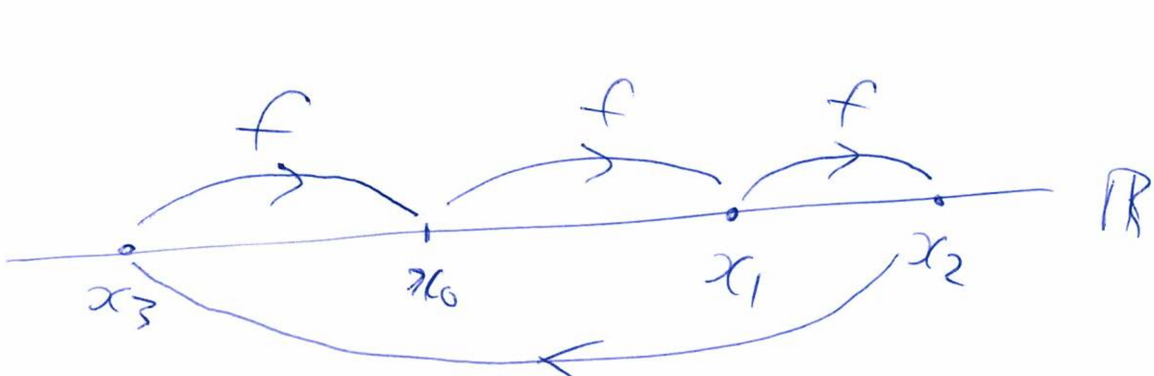
$\boxed{\text{i.e. } x_n = x_0}$

i.e. $\underbrace{f(f(f(\ldots f}_{n \text{ times}}(x_0)))) = x_0$

Schematic picture illustrating period 4



Picture illustrating period 4



<u>Warning</u> Here $f^n(x_0)$ denotes the n-fold composition $(\underbrace{f \circ f \circ \ldots \circ f}_{n \text{ times}})(x_0)$

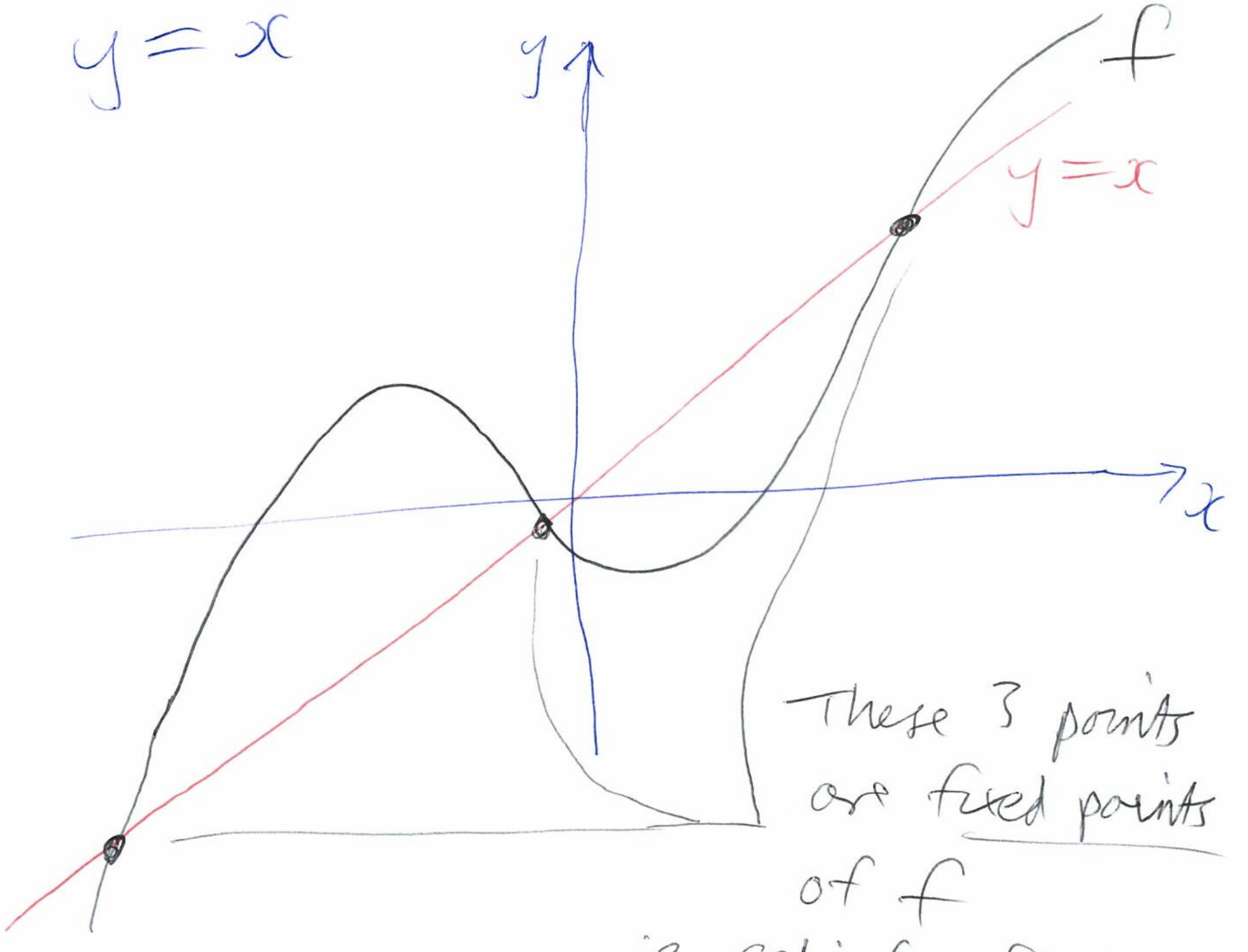
In particular, $f^n(x_0)$ does <u>NOT</u> mean that we "raise $f(x_0)$ to the n^{th} power", i.e. does <u>NOT</u> mean $(f(x_0))^n$

<u>ALSO</u> $f^n(x_0)$ does <u>NOT</u> mean the n^{th} derivative $f^{(n)}(x_0)$

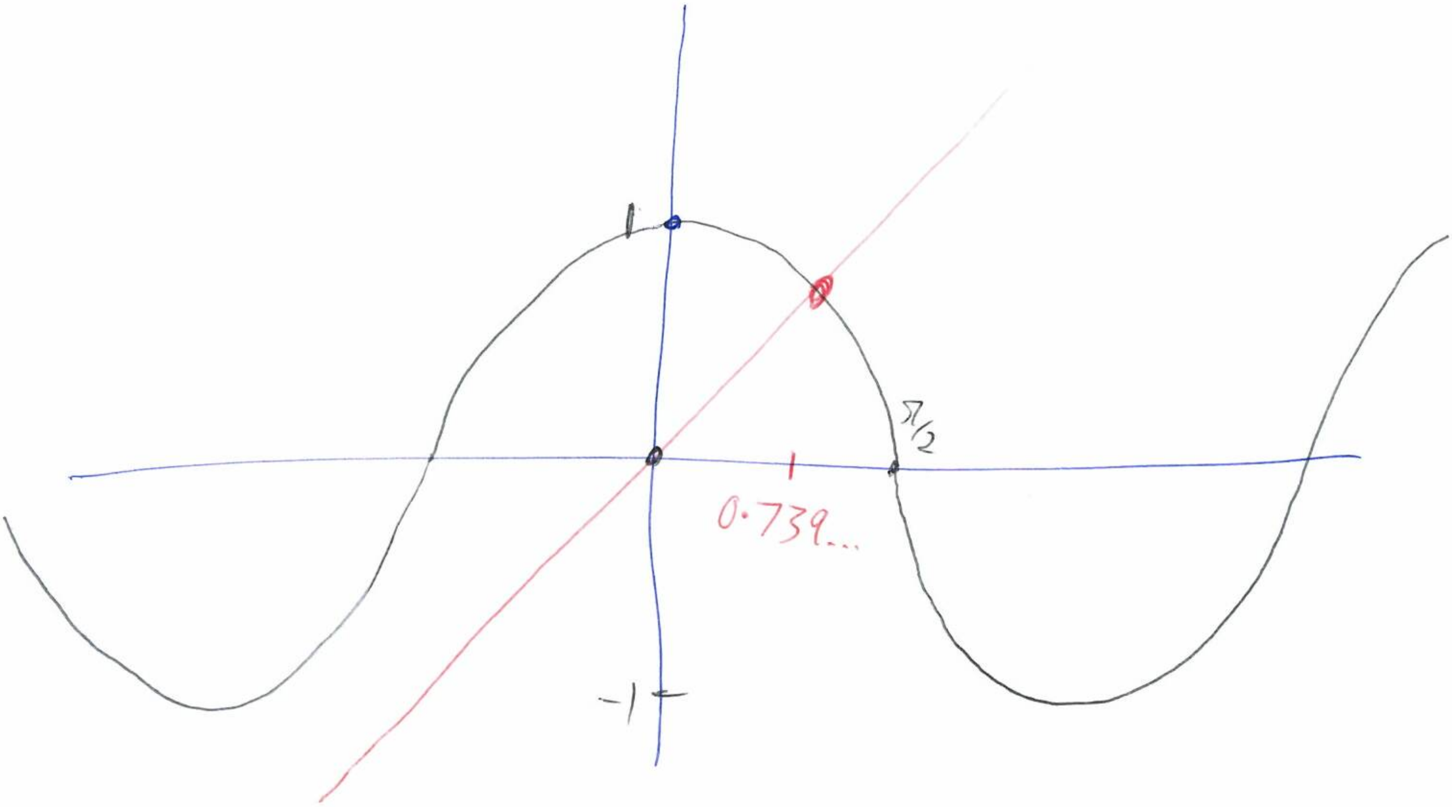
We denote the set of periodic points of period n by

$$Per_n(f) = \{x_0 \in \mathbb{R} : f^n(x_0) = x_0\}.$$

Geometrically we can think of fixed points as intersections of the graph of f with the line $y = x$



Example Recall $f(x) = \cos(x)$



we previously "saw" that the cosine function has fixed point at 0.739085...

<u>Defn</u> The <u>orbit</u> of $x_0 \in \mathbb{R}$ under $f : \mathbb{R} \to \mathbb{R}$ is the set

$$\mathcal{O}(x_0) = \{x_0, x_1, x_2, x_3, \ldots\}$$
$$= \{x_0, f(x_0), f^2(x_0), f^3(x_0), \ldots\}$$
$$= \{f^n(x_0) : n \geq 0\}$$

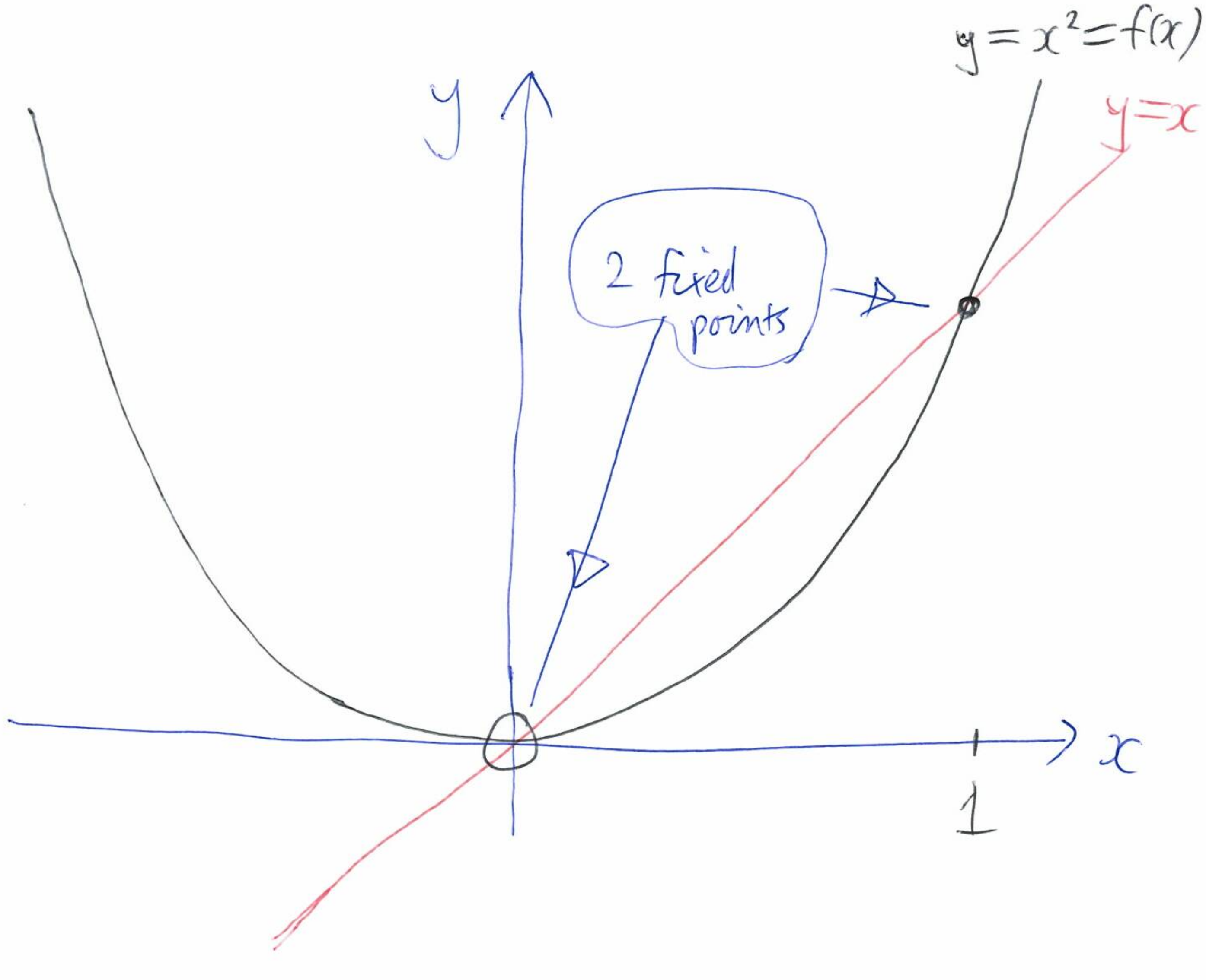
We usually consider the <u>orbit</u> of a point x_0 to be a <u>set</u> (as above), though occasionally it is useful to blur the distinction, and use 'orbit' to describe the <u>sequence</u> $(x_n)_{n=0}^{\infty}$.

<u>Example</u> Let us define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.

First, recall the recurrence relation

$$x_{n+1} = f(x_n)$$

$$\text{i.e. } x_{n+1} = x_n^2$$



This function f has 2 fixed points, namely 0 and 1.

(These are the solutions to the equation $x^2 = x$, and clearly they are the <u>only</u> solutions, i.e. the only fixed points).

The orbit of any point is given by

$$
\begin{array}{ccccccccc}
x_0 & \xrightarrow{f} & x_1 & \xrightarrow{f} & x_2 & \xrightarrow{f} & x_3 & \xrightarrow{f} & \cdots \\
 & & \parallel & & \parallel & & \parallel & & \\
 & & x_0^2 & & x_1^2 & & x_2^2 & & \\
 & & & & \parallel & & \parallel & & \\
 & & & & x_0^4 & & x_1^4 & & \\
 & & & & & & \parallel & & \\
 & & & & & & x_0^8 & &
\end{array}
$$

Observe that if $|x_0| > 1$ then the values of $x_n = f^n(x_0) = x_0^{2^n}$ will get large (in fact they tend to ∞ as $n \to \infty$)

e.g. $x_0 = 2$ then

$$2 \xrightarrow{f} 4 \xrightarrow{f} 16 \xrightarrow{f} \cdots$$

On the other hand, if $|x_0| < 1$ then the values of $x_n = f^n(x_0)$ will get smaller (in modulus), in fact they converge to 0 as $n \to \infty$.

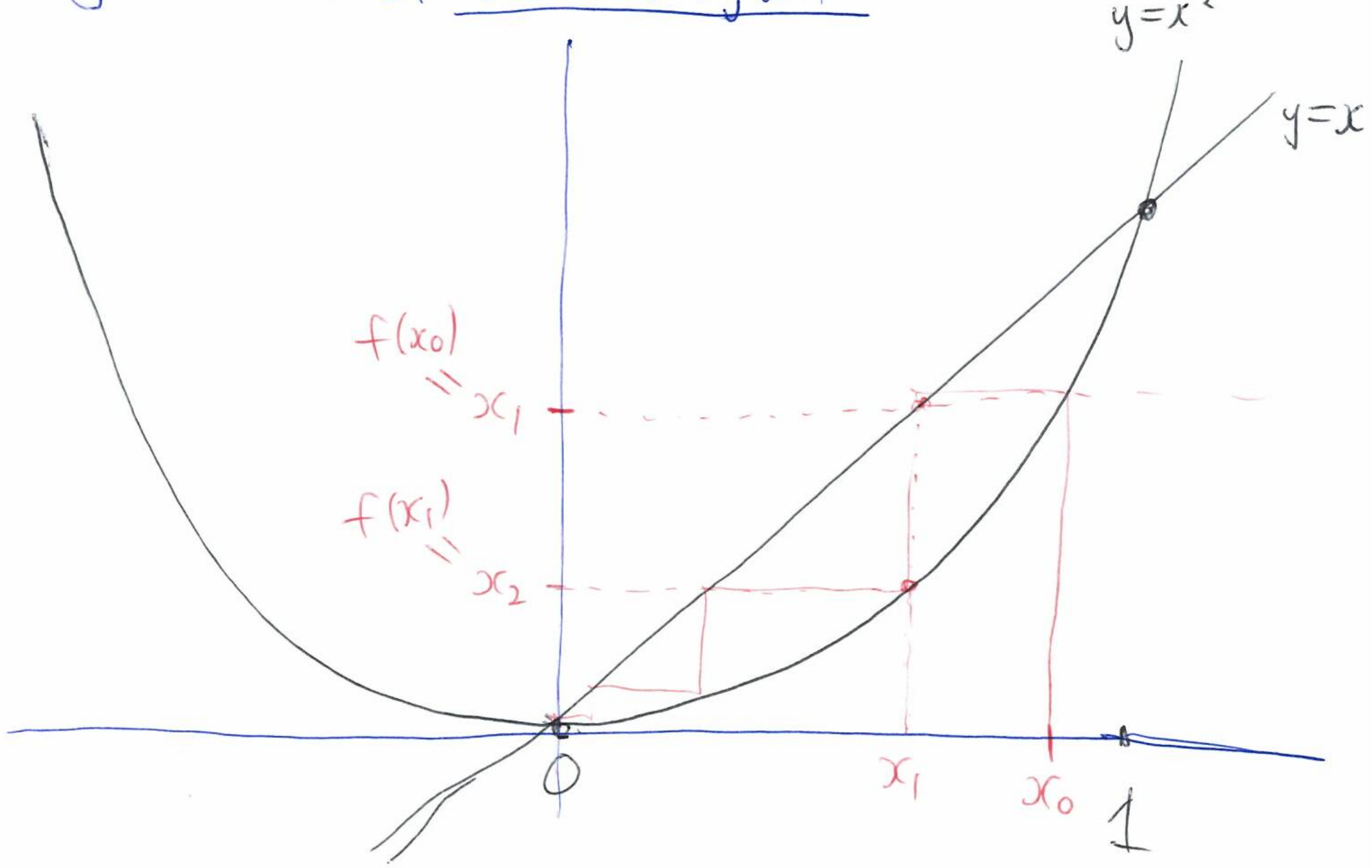
e.g. if $x_0 = 1/10 = 0.1$

$$0.1 \xrightarrow{f} 0.01 \xrightarrow{f} 0.0001 \xrightarrow{f} \cdots$$

Summary

$$\lim_{n\to\infty} f^n(x_0) = \begin{cases} 0 & \text{if } |x_0| < 1 \\ \infty & \text{if } |x_0| > 1 \\ 1 & \text{if } |x_0| = 1 \end{cases}$$

Returning to the graph of f, we can depict the orbit of x_0 using a so-called <u>cobweb diagram</u>



Note that the fixed point 0 is "attracting", in the sense that if $|x_0| < 1$ then the sequence $x_0, x_1, x_2, \ldots$ converges to the fixed point.

By contrast, the fixed point 1 is "repelling".

<u>Question</u> Does this $f(x) = x^2$ have any periodic points (apart from the two fixed points 0 and 1)?

<u>Answer</u> No! (Since the previous summary described all orbit behavior)

<u>Example</u> Define $f: \mathbb{R} \to \mathbb{R}$

by $f(x) = -x^3$

Examine fixed points and various orbits for this dynamical system f.

Recall the dynamical rule is given by

$$x_{n+1} = f(x_n) = -x_n^3$$