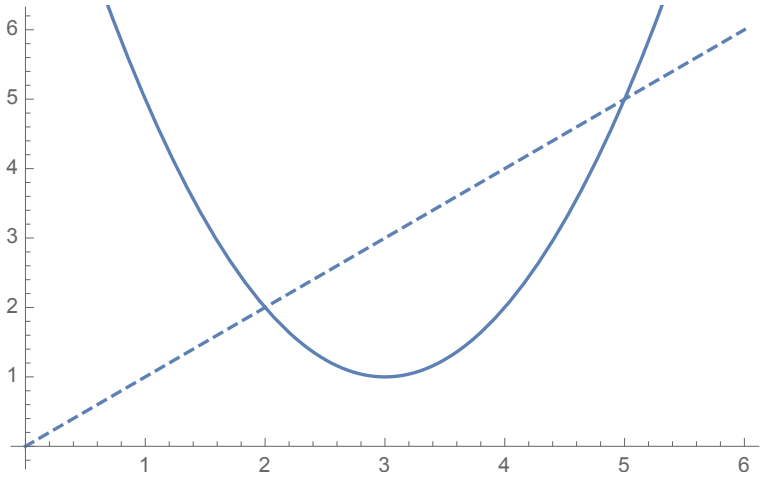
Chaos & Fractals

Solutions 1

(A) Suppose the map $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2 - 6x + 10$.

Exercise 1. Draw the graph of the map f, and determine all its fixed points. Determine which of these points are attracting and which of these points are repelling.

Fixed points satisfy $f(x) = x$, so $x^2 - 6x + 10 = x$, so $x^2 - 7x + 10 = 0$, so $x = 2$ and $x = 5$ are the fixed points.



Note that $f'(x) = 2x - 6$, so the multipliers at the fixed points are $f'(2) = -2$, and $f'(5) = 4$; both these multipliers have modulus strictly larger than 1, so both fixed points are *repelling*.

Exercise 2. For the map f, determine an eventually fixed point which is not a fixed point.

The point 1 is an example of an eventually fixed point (there are infinitely many other examples), since $f(1) = 5$.

Exercise 3. Draw a graph of the map f^2. Determine all the points of least period 2 of f. Determine which of these points are attracting and which of these points are repelling.

Points of period 2 satisfy $f^2(x) = x$, so

$$(x^2 - 6x + 10)^2 - 6(x^2 - 6x + 10) + 10 = x\,,$$

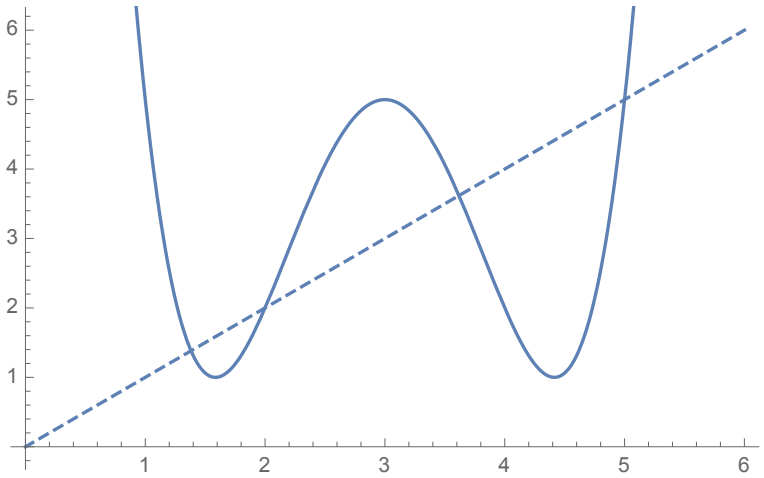
in other words

$$x^4 - 12x^3 + 50x^2 - 85x + 50 = 0\,.$$

We know that the two fixed points also have period 2, so we know that $x^2 - 7x + 10 = (x - 2)(x - 5)$ is a factor of the above righthand side, so we can factorise it to give the equation

$$(x - 2)(x - 5)(x^2 - 5x + 5) = 0\,,$$

so the two roots of $x^2 - 5x + 5$, namely $(5 \pm \sqrt{5})/2$, are points of least period 2.



The multiplier of the 2-cycle $\{(5 + \sqrt{5})/2, (5 - \sqrt{5})/2\}$ is equal to

$$f'((5 + \sqrt{5})/2) f'((5 - \sqrt{5})/2)\,,$$

and since $f'(x) = 2x - 6$ then $f'((5+\sqrt{5})/2) = -1+\sqrt{5}$ and $f'((5-\sqrt{5})/2) = -1-\sqrt{5}$ so

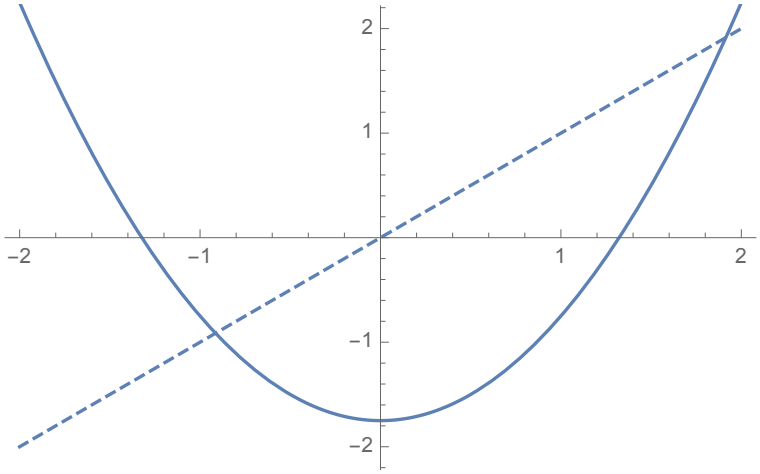
$$|f'((5 + \sqrt{5})/2) f'((5 - \sqrt{5})/2)| = |1 - 5| = |-4| > 1\,,$$

therefore the 2-cycle is *repelling*.

(B) Now suppose the map $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2 - 7/4$.

Exercise 4. Draw the graph of the map f, and determine all its fixed points. Determine which of these points are attracting and which of these points are repelling.

Fixed points satisfy $f(x) = x$, so $x^2 - x - 7/4 = 0$, so $x = (1 + \sqrt{8})/2$ and $x = (1 - \sqrt{8})/2$ are the two fixed points.



The multipliers at these fixed points are $f'((1 + \sqrt{8})/2) = 1 + \sqrt{8}$ and $f'((1 - \sqrt{8})/2) = 1 - \sqrt{8}$, both of which have modulus strictly larger than 1, so both fixed points are *repelling*.

Exercise 5. For the map f, determine an eventually fixed point which is not a fixed point.

One eventually fixed point is $-(1 + \sqrt{8})/2$, since $f(-(1 + \sqrt{8})/2) = (1 + \sqrt{8})/2$.

Exercise 6. Draw a graph of the map f^2. Determine all the points of least period 2 of f. Determine which of these points are attracting and which of these points are repelling.

Points of period 2 satisfy $f^2(x) = x$, so

$$(x^2 - 7/4)^2 - 7/4 = x\,,$$

which can be written as

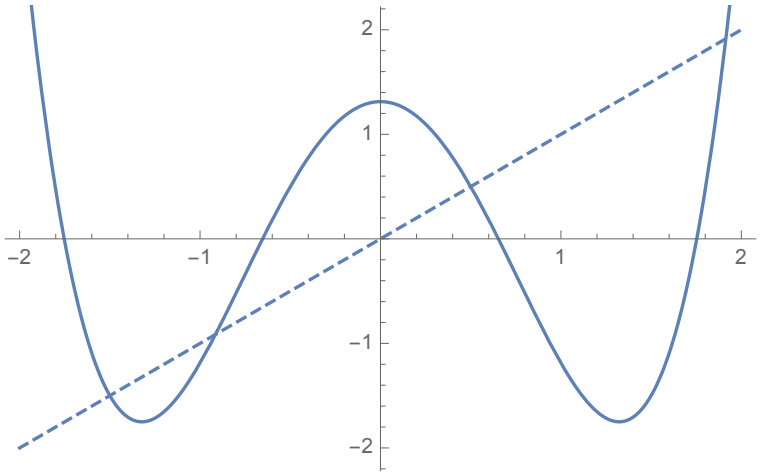
$$(x^2 - x - 7/4)(x - 1/2)(x + 3/2) = 0\,,$$

so the points of least period 2 are $1/2$ and $-3/2$.

Now $f'(1/2) = 1$ and $f'(-3/2) = -3$, so the multiplier for the 2-cycle $\{1/2, -3/2\}$ is

$$f'(1/2)f'(-3/2) = -3\,,$$

which is greater than 1 in modulus, so this 2-cycle is *repelling*.

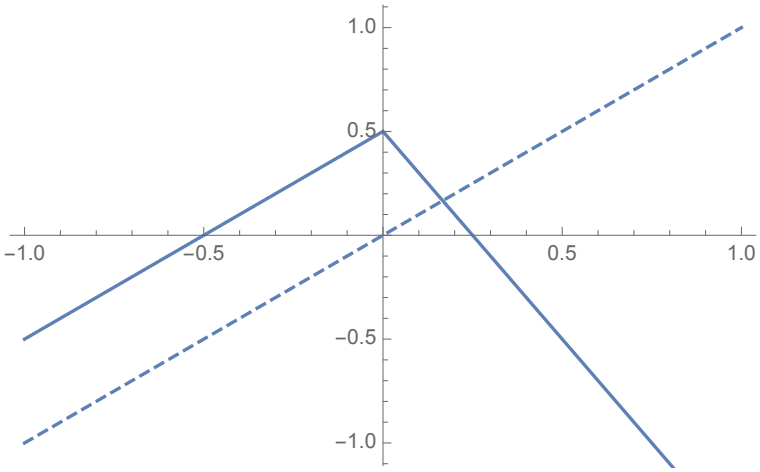


(C) Now suppose the map $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x + 1/2 & \text{for } x < 0 \\ -2x + 1/2 & \text{for } x \geq 0\,. \end{cases}$$

Exercise 7. Draw the graph of the map f, and determine all its fixed points. Determine which of these points are attracting and which of these points are repelling.

Fixed points satisfy $f(x) = x$. There are no such points $x < 0$, since $x + 1/2 = x$ has no solutions. There is one solution $x > 0$, namely the solution to $-2x + 1/2 = x$, namely $x = 1/6$ is the unique fixed point.



Since $|f'(1/6)| = |-2| = 2 > 1$ then this fixed point is *repelling*.

Exercise 8. For the map f, determine an eventually fixed point which is not a fixed point.

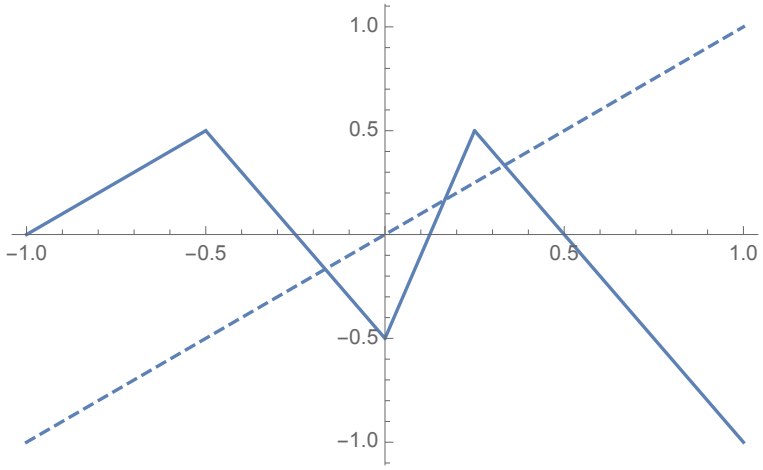
The point $-1/3 = 1/6 - 1/2$ satisfies $f(-1/3) = 1/6$, so $-1/3$ is an eventually fixed point. Note, for example, that more generally $1/6 - n/2$ is an eventually fixed point for all natural numbers n.

Exercise 9. Draw a graph of the map f^2. Determine all the points of least period 2 of f. Determine which of these points are attracting and which of these points are repelling.

Period-2 points x satisfy $f^2(x) = x$. If $x \leq -1/2$ then $f^2(x) = x + 1 \neq x$, so such an x cannot have period 2. If $x \in (-1/2, 0)$ then

$$f^2(x) = f(x + 1/2) = -2(x + 1/2) + 1/2 = -2x - 1/2\,,$$

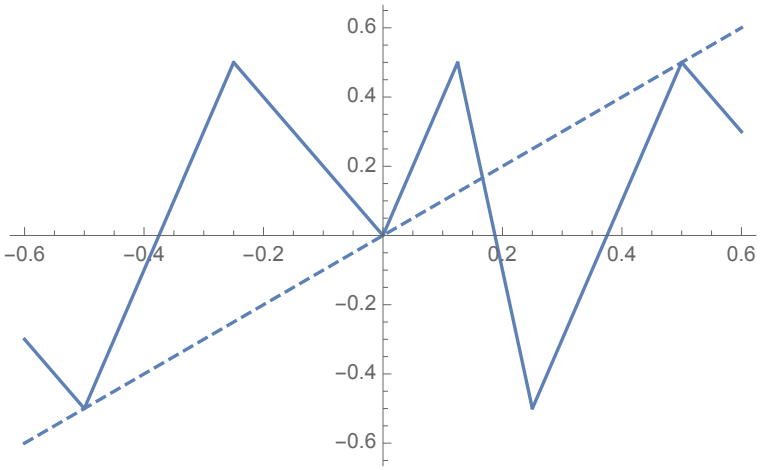
so $f^2(x) = x$ means $-1/2 = 3x$, so $x = -1/6$. Note that $f(-1/6) = 1/3$, so $\{-1/6, 1/3\}$ is a 2-cycle. In fact it is easily seen that it is the only 2-cycle (see e.g. that by inspection of the graph of f^2 there are 3 solutions to $f^2(x) = x$, namely the fixed point $1/6$, and the two period-2 points $-1/6$ and $1/3$).



The 2-cycle $\{-1/6, 1/3\}$ is *repelling*, since $|(f^2)'(-1/6)| = |-2| > 1$.

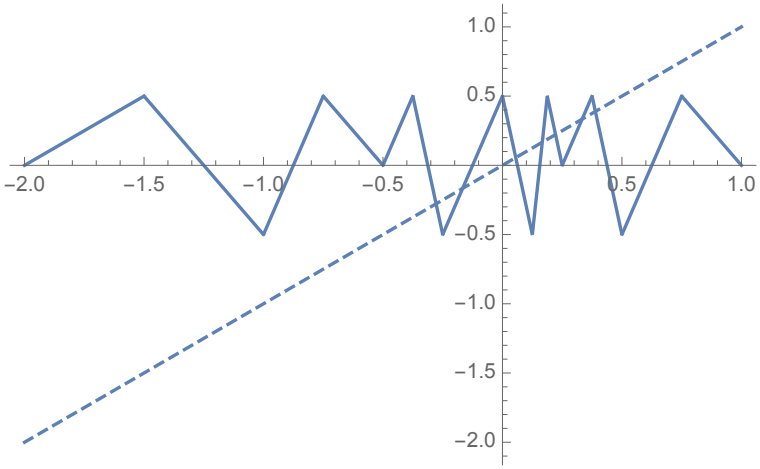
Exercise 10. For the map f, determine all its points of least period 3.

The points of period 3 are solutions to $f^3(x) = x$ (see below for a graph of f^3), and we can calculate that, in addition to the fixed point at $1/6$, the only such points are at $-1/2$, 0, and $1/2$, so these 3 points are of least period 3, and constitute a 3-cycle.



Exercise 11. For the map f, determine all its points of least period 4.

The points of period 4 are solutions to $f^4(x) = x$ (see below for a graph of f^4), and we can calculate that, in addition to the fixed point at $1/6$, and the points $-1/6, 1/3$ of least period 2, the only such points are at $-5/18$, $1/18$, $2/9$, and $7/18$, so these 4 points are of least period 4, and constitute a 4-cycle.



Exercise 12. Is it the case that f has a point of least period n for every $n \in \mathbb{N}$?

Indeed f has a point of least period n for every $n \in \mathbb{N}$, by Sharkovskii's Theorem, since f is continuous and has a point of least period 3.

Exercise 13. Can you guess (or even prove) a formula for the number of points of period n for the map f?

Let P_n denote the number of points of period n. Exercise 7 gives $P_1 = 1$, exercise 9 gives $P_2 = 1 + 2 = 3$, exercise 10 gives $P_3 = 1 + 3 = 4$, and exercise 11 gives $P_4 = 1 + 2 + 4 = 7$.

These exercises may *suggest* that there are n points of least period n for each $n \in \mathbb{N}$ (and hence that P_n equals the sum of the divisors of n). However this is *false*, and it fails at $n = 5$: you can check that there are *two* orbits of least period 5, so 10 points of least period 5, therefore $P_5 = 1 + 10 = 11$ (the 11 solutions to the equation $f^5(x) = x$ are visible in the graph of f^5 below).

In fact the sequence $(P_n)_{n \geq 1}$ begins as $1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \ldots$, and it can be shown that P_n is the solution of the difference equation $P_{n+2} = P_n + P_{n+1}$ with initial values $P_1 = 1$, $P_2 = 3$ (this is reminiscent of the construction of the Fibonacci sequence). An alternative formula for P_n is as the trace (the sum of the diagonal entries) of the n^{th} power of the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.