Pre-Sessional course

Mathematics for Macroeconomics

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Aim of the module

- To provide the mathematical tools necessary for understanding modern macroeconomics.
- To develop intuition for applying these tools to dynamic models.
- To strengthen confidence in linking formal methods with economic reasoning.

Teaching Schedule

Hour	Slides	Topics
Hour 1	1–50	Introduction, Linear systems: definitions and examples, Eigenvalues, Quadratic forms, Trigonometric functions, Complex numbers
Hour 2		Continuous Time Dynamics I: differential equations basics, first-order DEs (separable, linear), qualitative vs analytical solutions, stability of first-order equations
Hour 3		Continuous Time Dynamics II: higher-order equations, oscillatory solutions (damped vs explosive cycles), linear systems in matrix form, trace–determinant rule
Hour 4		Continuous Time Dynamics III: phase diagrams for 2D systems, nonlinear systems, saddle points (theory, macro example K–C, linear system example)
Hour 5		Optimal Control (intro) + Discrete Time: state/control variables, Hamiltonian, maximum principle, infinite horizon, transversality. Discrete time: difference equations, stability, stochastic difference equations (Blanchard–Khan)

Readings

- D. Acemoglu, Introduction to Modern economic growth", Princeton University Press
- A. Chiang, "Fundamental methods in mathematical economics", McGrawHill
- A. Chiang, "Elements of dynamic optimization", McGrawHill
- J. Miao, "Economic dynamics in discrete time", MIT Press
- K. Sydsaeter et al., "Further mathematics for economic analysis", FT Prentice Hall

Note: PDF copies of these books are saved in the folder “readings” on QMPLUS for your reference

TOPIC 1

1. Linear systems: definitions and examples
2. Eigenvalues
3. Quadratic forms
4. Trigonometric functions
5. Complex numbers

1.1 LINEAR SYSTEMS: DEFINITIONS AND EXAMPLES

1. What is a Linear System?

- A linear system is a set of linear equations.
- Each equation represents a line, plane, or hyperplane.
- Example:

$$\begin{cases} 2x + 3y = 8 \\ -x + 4y = 1 \end{cases}$$

Goal: Find values of x and y that satisfy all equations.

What Is a Matrix?

- A matrix is a rectangular array of numbers.
- We can write linear systems using matrix notation: $Ax = b$, where:
- A is the coefficient matrix
- x is the vector of unknowns
- b is the vector of constants

Solving a Linear System

- Solving means finding values of x that make Ax = b true.
- Three possible outcomes:

1. One solution
2. Infinitely many solutions
3. No solution

Square Matrix Case (m = n)

- If A is square and $|A| \neq 0$: Unique solution exists.
- If $|A| = 0$: Either no solution or infinitely many.
- Determinant tells us if the matrix is invertible (if $\neq 0$).

Rank and Solutions

Rank: Number of independent rows (or columns).

- If rank(A) = rank(A|b): At least one solution.
- If rank(A) < rank(A|b): No solution.
- If rank(A) < number of variables: Infinitely many solutions.

- See notes or next slide for rank(A|b)

Note on the side

In linear algebra, rank(A|b) refers to the rank of the augmented matrix [A|b], which is formed by combining the coefficient matrix A with the column vector b. The rank of a matrix is the number of <u>linearly independent rows (or columns) in the matrix</u>. The rank of the augmented matrix [A|b] is crucial for determining whether a system of linear equations represented by Ax = b has a solution.

Augmented Matrix:

- The augmented matrix [A|b] is created by appending the column vector 'b' (the constants from the right side of the equations) to the coefficient matrix 'A'.

Rank:

- The rank of a matrix is the maximum number of linearly independent rows (or columns).

System of Equations:

The equation Ax = b represents a system of linear equations, where A is the coefficient matrix, x is the vector of unknowns, and b is the constant vector.

Consistency:

- A system of equations is considered consistent (meaning it has at least one solution) if and only if the rank of the original matrix A is equal to the rank of the augmented matrix [A|b]. If these ranks are not equal, the system is inconsistent and has no solutions.
- In simpler terms: If adding the column 'b' to matrix 'A' doesn't change the "essential dimensionality" of the system (as measured by rank), then there's a solution. If it does change the dimensionality, there's no solution

Analogy: Equations as Clues

- Each equation = a clue.
- Independent clues help solve the mystery.
- Redundant clues = no new info.
- Contradictory clues = no solution.

So, your original system:

- $A(m \times n) \cdot x(n \times 1) = b(m \times 1) A(m \times n) \cdot x(n \times 1) = b(m \times 1)$

Means:

- You have m equations and n unknowns.
- You're looking for values of x that satisfy all equations.
- Whether you have a unique solution, no solution, or infinitely many solutions depends on **how independent** the equations are — this is what **determinants** and **rank** help us measure.

Solving $Ax = b$: the Rank Test (what A and $b$mean) What are A and b?

A: coefficient matrix (numbers multiplying the variables).
b: right-hand-side constants.
Augmented matrix $(A \mid b)$:matrix A with b appended as an extra column.

Rank facts (Rouché–Capelli):
rank$(A) = rank(A \mid b) = n$ → **one unique solution** (full pivots).
rank$(A) = rank(A \mid b) < n$ → **infinitely many solutions** (free variables).
rank$(A) < rank(A \mid b)$ → **no solution** (inconsistent).

Geometric intuition:
Columns of A span all reachable right-hand sides.
If b lies **in** that span → solvable; if **not** → no solution.

Cont. for example and graphic intuition

Examples

1. Unique solution: $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

R2*←R2−R1: $\begin{bmatrix} 1 & 1|2 \\ 0 & -2|0 \end{bmatrix}$

rank(A) = $rank(A \mid b) = 2 = n \Rightarrow$ **unique solution**

2. No solution: $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Row-reduce $(A \mid b) \rightarrow \begin{bmatrix} 1 & 1 \mid 1 \\ 0 & 0 \mid 1 \end{bmatrix} \Rightarrow$ contradiction ⇒ ranks rank(A)=1 ≠ rank(A|b)=2 ⇒ **no solution**.

3. Infinitely many solutions $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

R2*←R2−2R1 $(A \mid b) \rightarrow \begin{bmatrix} 1 & 1 \mid 1 \\ 0 & 0 \mid 0 \end{bmatrix}$,

rank(A) = $rank(A \mid b) = 1 < n = 2 \Rightarrow$ **infinitely many**

R2* is the transformed, ”new” R2

Graphically

Goal:

- Show how equations in two or three variables correspond to lines or planes, and how their **intersections represent solutions**.

Two Variables (2D)

- Each equation is a **line**.

Show:

- **One solution**: lines intersect at a point.
- **No solution**: lines are parallel.
- **Infinite solutions**: lines coincide.

Tools:

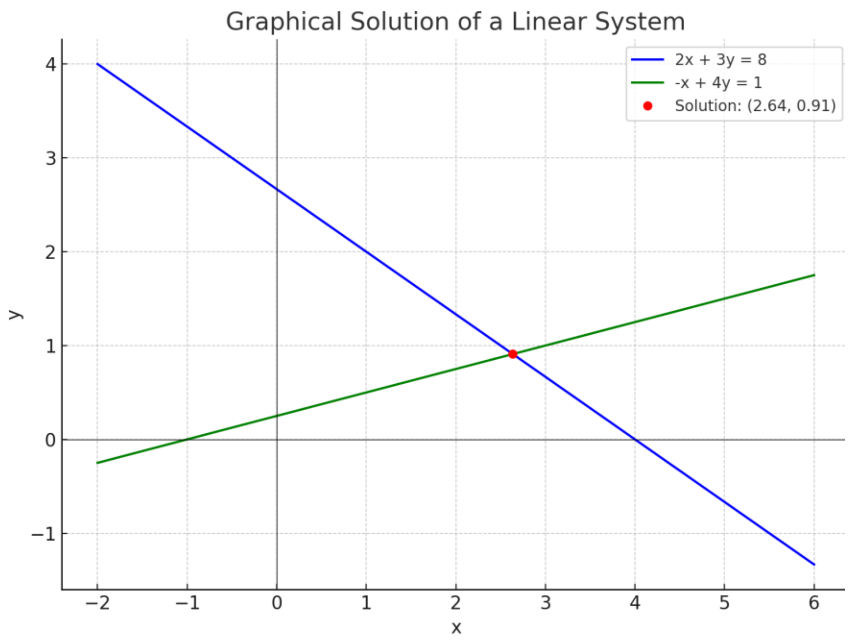
Desmos (https://www.desmos.com)

GeoGebra

Python with matplotlib

Example:

- Plot both lines. The intersection is the unique solution.



Intuitively

why solutions exist (or don’t) in terms of **information and constraints**.

Analogy: Solving a Mystery

Each equation is a **clue**.

More clues can help narrow down the solution — but only if they’re independent.

Redundant or conflicting clues either don’t help or create a contradiction.

Dimensions & Geometry

If you have **2 unknowns**, the solution is a point in 2D space.

1 equation → a line (too many possibilities),

2 equations → possibly a point (just right),

3 or more → may over-constrain the system.

Degrees of Freedom

If the system has **free variables**, that means you have **choice** in the solution — a **family** of solutions.

Examples from Economics and Finance

Example 1: Budget Constraint

- A consumer has a budget of £100:
- 2x+4y=1002x+4y=100xx: number of goods A (price £2),
- y: number of goods B (price £4).
- This is a line in 2D: all the combinations of goods they can afford.

Example 2: Leontief Input-Output Model. Model: $x = Ax + b$

- In matrix form:
- input coefficients (what each sector needs from others),
- x: total output levels,
- b: final demand vector.
- You solve this to find **how much each sector must produce**.
- Interpretation: Ax = intermediate demand; b = final demand

Solve: $(I - A)x = b \Rightarrow x = \left(I - A\right)^{-1}b)$ Leontief inverse, if $I - A$ is invertible)

Objects:

- A: input coefficients
- x: gross outputs
- b: final demand

Example 3 Arbitrage Pricing Theory (APT) as a Linear System (page 1 of 2)

$$r_i = \beta_{i1} f_1 + \beta_{i2} f_2 + \cdots \beta_{in} f_n + \epsilon_i$$

Where:

- r_i = return on asset i
- β_{ij} = sensitivity (or "loading") of asset i to factor j
- f_j = return on factor j
- ϵ_i = idiosyncratic (asset-specific) shock

Matrix Form

- Suppose we observe m assets and assume n factors affect returns. Then:
- $$\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m1} & \beta_{m2} & \cdots & \beta_{mn} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_m \end{bmatrix}$$

That is:

$$r = Bf + \epsilon$$

Where:

- r = vector of observed returns ($m \times 1$)
- B = factor loadings ($m \times n$)
- f = vector of unknown factors ($n \times 1$)
- ϵ = vector of errors ($m \times 1$)

Solving and Interpreting the APT System (page 2 of 2)

Linear System Setup

- This is a linear system with:
- Known: r, the asset returns
- Known: B, the matrix of factor loadings (from regression)
- Unknown: f, the factor returns

Solving for the Factors f

- If we assume the error term is small or zero on average, we can estimate factor returns f by solving:

$$Bf \approx r$$

- This is often done via **least squares**:

$$\hat{f} = \left(B^T B\right)^{-1} B^T r$$

- This is the solution to a linear system that **minimizes the squared errors** — very common in econometrics.

Economic Interpretation: This setup is very useful in finance for:

- Estimating **systematic risk exposures**,
- Finding **common drivers of asset returns**,
- Constructing **hedged portfolios**,
- Understanding **macroeconomic sensitivity** of assets.

For example, the factors might include:

- Interest rate changes,
- Inflation shocks,
- Oil price movements,
- Market index returns.

Component	Linear System Term	Interpretation
r	known output vector	Observed returns of assets
B	known coefficient matrix	Factor sensitivities (betas)
f	unknown vector	Factor returns to estimate
ϵ	residual	Unexplained (idiosyncratic) return

1.2 EIGENVALUES

2. What Are Eigenvalues?

Why Are We Talking About Eigenvalues?

In economics, many models describe how things change **over time**. For example:

- How output or capital evolve in the future
- How a system reacts to a shock
- Whether things **stabilise**, **grow**, or **explode**

Often, these models take the form:

$x_{t+1}=Ax_t$

Where:

- x_t is a vector of variables (e.g., output and capital)
- A is a matrix of relationships or coefficients

To **predict what happens over time**, we need to understand what the matrix A does. That's where **eigenvalues** come in.

Background intuition

A vector has two features: a **direction** and a **magnitude** (length).

Example:

v = $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (a column vector). This means:

- Move 1 step to the right
- Move 2 steps up

Intuition: it’s an arrow pointing diagonally upward.

A matrix is a **rule** (linear transformation) that acts on vectors.

It can:

- Stretch them
- Shrink them
- Rotate them
- Flip (reflect) them

We say we “apply the matrix to the vector” — i.e., we multiply them: Av

What Happens When You Multiply?

Example

Let

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

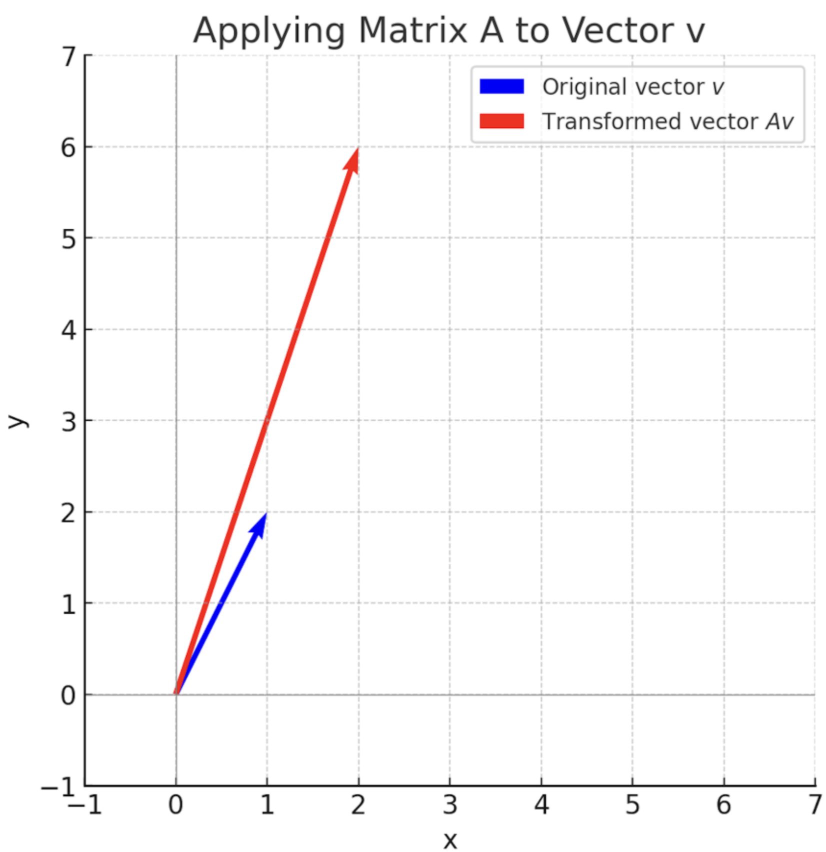
Then:

$$Av = \begin{bmatrix} 2 \cdot 1 + 0 \cdot 2 \\ 0 \cdot 1 + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

Interpretation:

The vector becomes **longer and steeper** — its **direction and length have changed** after multiplication by A.

Graphically



Graphic Example: What Happens When You Multiply?

- The **blue arrow** is the original vector

$$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- The **red arrow** is the result of applying the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

which **stretches** the vector:

Doubles the x-component

Triples the y-component

The transformation changes both the **length** and the **direction** of the vector (it becomes steeper).

…But some vectors are special

Some vectors are **special**: when you apply the matrix to them, their direction **does not change**.

They only get **stretched** or **shrunk:** In other words: $Av=\lambda v$

- v is an **eigenvector** — a direction that stays the same
- λ is the **eigenvalue** — it tells you how much the vector is stretched, shrunk or flipped (direction does not change, orientation is reversed)

Value of λ	What It Does
$\lambda > 1$	Vector is stretched
$0 < \lambda < 1$	Vector is shrunk
$\lambda = 1$	Vector stays the same
$\lambda = -1$	Vector flips direction
$\lambda = 0$	Vector collapses to zero

How Do You Find Eigenvalues?

You use this formula:

det(A−λ I)=0

The **solutions** to this equation are the **eigenvalues**.

$A - \lambda I$ means: subtract the scalar λ from the **diagonal** of A.

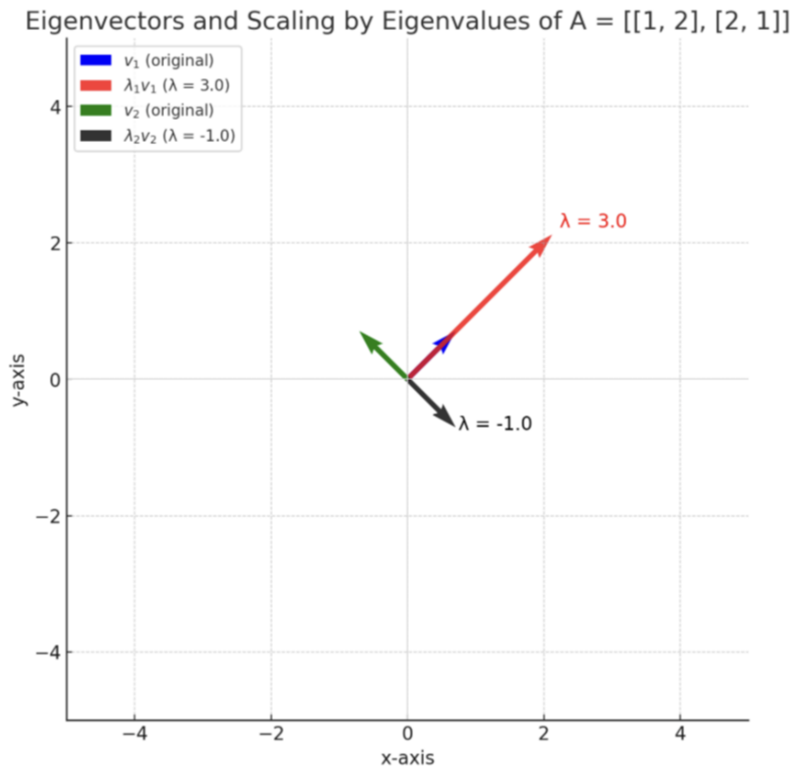
- Det $(A - \lambda I) = 0$ is the **characteristic equation**; its solutions λ are the **eigenvalues**.
- Quick example:

A=$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$, $A - \lambda I$=$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix}$

Det(A−λI)=(2−λ)(3−λ)=0 ⇒. λ=2,3.

Where I is the identity matrix

The direction remains unchanged, with all arrows aligned along the same line as their corresponding partners. The length of each vector is scaled by its eigenvalue, which can also invert the vector if the eigenvalue is negative (e.g., three times longer or flipped).



$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Step-by-step calculation of eigenvalues and eigenvectors:

Step 1: Start from the eigenvalue equation

$Av=\lambda v \Rightarrow (A-\lambda I)v=0$

Step 2: Set up the characteristic equation

$\det(A-\lambda I)=0 \Rightarrow \det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} =0$

Step 3: Compute the determinant

$(1-\lambda)^2-4=\lambda^2-2\lambda-3=0.$

Step 4: Solve the quadratic

$\lambda^2-2\lambda-3=0. \Rightarrow. \quad \lambda=3,-1$

These are the **eigenvalues**:

$\lambda_1=3$

$\lambda_2=-1$

Find eigenvectors

For λ=3:

- Solve (A−3I)v=0⇒ $\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$v=0

This gives:

- v_1= $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (or any multiple of it)

For λ=−1:

- Solve (A−I)v=0⇒ $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$v=0

This gives:

- v_2= $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (or any multiple of it)

Let's have an example

Imagine a **two-variable dynamic system**:

$$\begin{bmatrix} Y_{t+1} \\ K_{t+1} \end{bmatrix} = A \begin{bmatrix} Y_t \\ K_t \end{bmatrix} \quad \text{where} \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Y = output

K = capital

This system could represent a **stylised multiplier process**, where: Output depends on capital, Capital depends on output

Shocks to one variable amplify through their interaction

- $\lambda_1=3$: this direction **explodes** (unstable)
- $\lambda_2=-1$: this direction **oscillates**

So the system is **not stable** — it either **grows explosively** or **flips signs** each period depending on the initial condition.

1.3 QUADRATIC FORM

Quadratic form

A **quadratic form** is a scalar expression of the form:

$Q(x)=x'Ax$. Where:

- x is a column vector of size $n\times 1$
- x' is its transpose (a $1\times n$ row vector)
- A is an $n\times n$ matrix

So the multiplication:

- Ax → gives an $n\times 1$ vector
- $x'(Ax)$ → gives a scalar (a number)

Interpreting Q(x)

Q(x) tells us how the vector x interacts with the matrix A.

Why is this relevant?

It appears in:

- Optimization problems
- Stability analysis
- …

Positive/Negative (Semi)Definiteness

We classify the **matrix A** based on the value of $Q(x)$ for all $x \neq 0$

Matrix Type	**Condition on** $Q(x)$
Positive definite	$Q(x) > 0$
Positive semidefinite	$Q(x) \geq 0$
Negative definite	$Q(x) < 0$
Negative semidefinite	$Q(x) \leq 0$
Indefinite	$Q(x)$ takes both signs

Link to Eigenvalues

Under symmetry, we don't need to check Q(x) for all x:
We just look at the **eigenvalues λ_i** of A

Matrix Type	Condition on eigenvalues
Positive definite	All $\lambda_i > 0$
Positive semidefinite	All $\lambda_i \geq 0$
Negative definite	All $\lambda_i < 0$
Negative semidefinite	All $\lambda_i \leq 0$
Indefinite	Some $\lambda_i > 0$, some < 0

Example : Deviations from a Policy Target page

Imagine a central bank minimizing a **quadratic loss function** that penalizes deviations from policy targets.

Let: $x = \begin{bmatrix} \pi \\ y \end{bmatrix}$ where:

- π is the deviation of inflation from target
- y is the deviation of output from potential output

Define the loss function:

$L(x)=x'Ax$

Where A is a symmetric, positive definite matrix encoding how severely the central bank penalizes each type of deviation. Cont.

Example : Deviations from a Policy Target

We choose the matrix:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

let the vector of deviations be

$$x = \begin{bmatrix} \pi \\ y \end{bmatrix}$$

where:

- π = inflation deviation from target
- y = output deviation from potential

Step 1: Apply the linear transformation

$$L(x) = x^T A x$$
$$L(x) = \begin{bmatrix} \pi & y \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \pi \\ y \end{bmatrix}$$

Step 2: Multiply out

First multiply A and x:

$$Ax = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \pi \\ y \end{bmatrix} = \begin{bmatrix} 2\pi + y \\ \pi + 3y \end{bmatrix}$$

Then multiply by x^T:

$$L(x) = \begin{bmatrix} \pi & y \end{bmatrix} \begin{bmatrix} 2\pi + y \\ \pi + 3y \end{bmatrix}$$
$$L(x) = \pi(2\pi + y) + y(\pi + 3y)$$

Step 3: Simplify

$$L(x) = 2\pi^2 + 2\pi y + 3y^2$$

Interpretation

- $2\pi^2$:penalty for inflation deviation (quadratic cost, grows quickly with size of deviation)
- $3y^2$:penalty for output deviation (even larger weight here)
- $2\pi y$: interaction term (inflation and output deviations jointly costly)

This is like a quadratic loss function in policy models: it tells us how far the economy is from its targets, weighting inflation and output.

cont.

Economic Interpretation

This is a **weighted penalty**:

- Deviations in inflation and output both **increase the loss**.
- The cross term $2\pi y$ says that **simultaneous deviations** in both inflation and output are even worse than each alone.

The matrix A being **positive definite** means:

- $L(x)>0$ for all $x \neq 0$: **all deviations are costly**.
- There's a **unique minimum at x=0**— that is, when inflation is on target and output is at potential.
- The central bank has **well-defined preferences** and **no direction** in which deviations would be "rewarded" or costless.

Food for thought

Why Use a Quadratic Form?

- **Symmetric treatment** of deviations (inflation and output)
- **Mathematically convenient**: convex, differentiable, easy to optimise
- **Policy interpretation**: might reflect trade-offs (e.g. inflation bias vs output gap)

1.4 TRIGONOMETRIC FUNCTIONS

Definition from the Unit Circle

On the unit circle (radius = 1):

Any point (u,v) on the circle can be reached by moving an angle x radians from the point (1,0) counterclockwise.

Then:

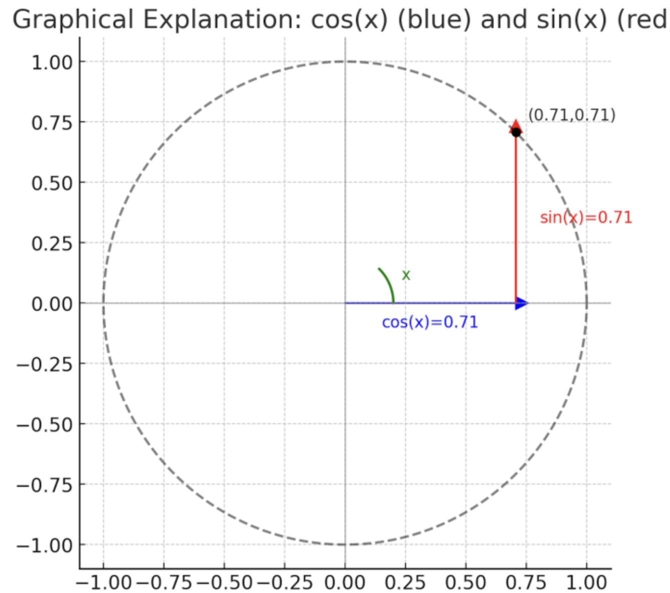
u=cos(x)

v=sin (x)

So:

cos(x) is the **horizontal coordinate**

sin(x) is the **vertical coordinate**



Graph Intuition

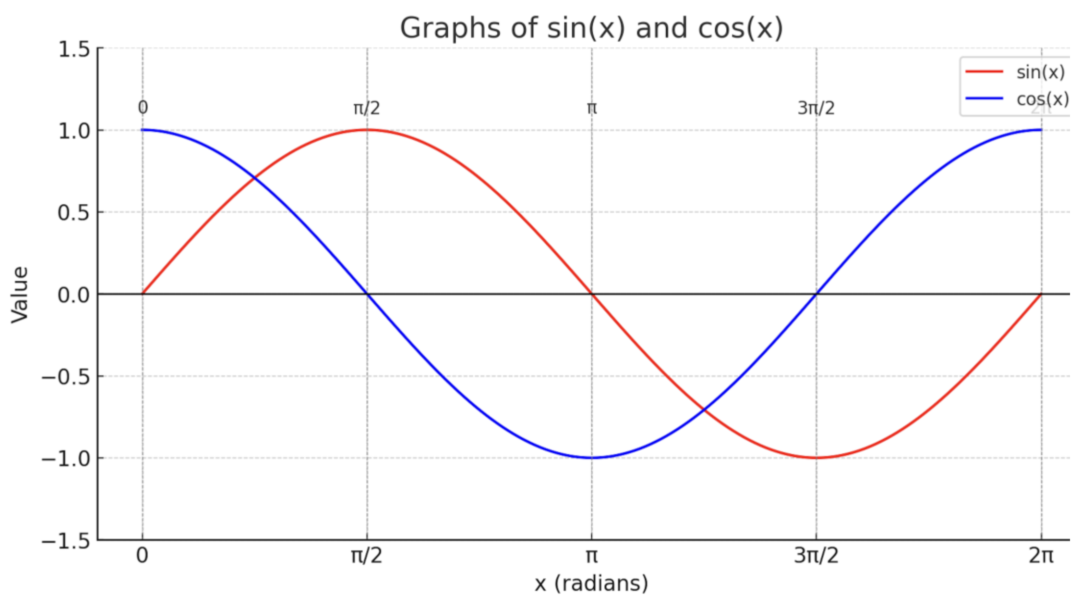
From the plot

Sine (red):

- Starts at 0
- Rises to 1 at π/2π/2
- Back to 0 at ππ
- Down to −1 at 3π/23π/2
- Returns to 0 at 2π2π

Cosine (blue):

- Starts at 1
- Falls to 0 at π/2π/2
- Down to −1 at ππ
- Back to 0 at 3π/23π/2
- Returns to 1 at 2π2π



Understanding Radians

- A radian measures **arc length** on a circle of radius 1.
- Since the whole circle has circumference 2π, one full turn is:
- 2π radians =360∘
- Quarter turn → π/2,   Half turn → π, etc.

Angle x (radians)	$\cos(x)$	$\sin(x)$	Interpretation
0	1	0	Start on x-axis
$\pi/2$	0	1	Top of the circle
π	–1	0	Leftmost point
$3\pi/2$	0	–1	Bottom of the circle
2π	1	0	Back to start

Real-World Applications

- **DSGE[1] models** (e.g. in RBC[2] theory):
  Stylized output gap often approximated with sinusoidal shocks.
- **Seasonal adjustments** in macro data:
  Many statistical agencies remove **sinusoidal seasonal effects** (e.g. Christmas spikes in consumption).
- **Harrod-Domar instability**:
  If growth doesn’t match savings/investment equilibrium, output may oscillate.

1 Dynamic Stochastic General Equilibrium
2 Real Business Cycle

Analogy: Central Bank Overreaction and Correction

Scenario:
A central bank is trying to **stabilize inflation** and **output** using interest rates. But it only observes the economy with **a delay**, and reacts based on **imperfect forecasts**.

What Happens?
Shock hits the economy: e.g. a cost-push shock raises inflation.
The central bank **tightens policy** — raises interest rates.
But by the time the effect kicks in, inflation has already begun to fall on its own.
The **delayed effect** of high rates **pushes inflation too far down**.
The central bank sees low inflation → now **cuts rates**.
Again, too late — and inflation starts rising again.

Result:
You get an **overshooting-and-correction cycle** — not because of irrationality, but because of:
- Reaction lags
- Intertemporal decision-making
- Forward-looking expectations

These cyclical movements — **above and below target** — are naturally and **mathematically captured by sine and cosine functions**.

Trigonometric Functions Capture:
The **timing** (phase)
The **amplitude** of deviation
The **regularity** of cyclical dynamics in response to policy feedback

Function	Period (Wavelength)	Amplitude
$y = \sin x$	2π	1
$y = \sin(ax)$	$\frac{2\pi}{a}$	1
$y = A\sin(ax)$	$\frac{2\pi}{a}$	A
$y = A\sin(ax) + B$	$\frac{2\pi}{a}$	A *(midline shifted to* B*)*

Frequency is the **inverse of period**:
Frequency=a/2π(number of oscillations per radian)

1.5 COMPLEX NUMBERS

Topic 1.5 Complex Numbers: What Are They?

A **complex number** is written as: $z=a+bi$

- Where:

a is the **real part**

b is the **imaginary part**

$i=-1$is the **imaginary unit**

Why Do We Use Them?

- Allow us to solve equations like:
- $x^2+1=0 \Rightarrow x=\pm i$
- Essential for finding **all solutions** to:
  - Quadratic equations
  - Differential equations
  - Characteristic equations in macro models

In Macroeconomics?

Complex numbers arise:

- When solving **dynamic systems**
- When **eigenvalues** are complex (→ oscillatory behaviour)