Maths & Stats Pre-Sessional Tutorial

Topic 5: Basic of Derivatives and Matrix Algebra - Solutions

Exercise 1

Find the first derivative of the following functions:

a) $x^2 + 3x + 1$

Solution: $f'(x) = 2x + 3$

b) $(2x - 3)^5$

Solution: General Rule: This is a case of $h(x) = f(g(x))$, where $g(x) = 2x - 3$ so :

$$h'^{(x)} = f'\big(g(x)\big) * g'(x) =$$

$$5(2x - 3)^4 * 2 =$$

$$10(2x - 3)^4$$

c) $\frac{e^x}{x-1}$

Solution: General Rule: This is a case of $h(x) = \frac{f(x)}{g(x)}$. We know that

$h'^{(x)} = \frac{f'(x)g(x) - f(x)g'^{(x)}}{g(x)^2}$, so:

$$h'(x) = \frac{e^x(x-1) - e^x}{(x-1)^2}$$

d) $4x^3(2x - 1)$

Solution: General Rule: This is a case of $h(x) = f(x)g(x)$. We know that

$h'^{(x)} = f'(x)g(x) + f(x)g'(x)$. Therefore,

$$h'(x) = 12x^2(2x - 1) + 4x^3(2) =$$

$$24x^3 - 12x^2 + 8x^3$$

$$32x^3 - 12x^2$$

e) $\ln(3x)$

Solution: General Rule: This is a case of $h(x) = f(g(x))$, where $g(x) = 3x$ so :

$$h'^{(x)} = f'\big(g(x)\big) * g'(x) =$$

$$\frac{1}{3x} * 3 = \frac{1}{x}$$

Exercise 2

Are functions (a), (b) and (e) in Exercise 1 concave or convex? Why?

Solution: In order to answer this question, we need the second derivative of each function.

- Function (a) → $f''(x) = 2$ so this is strictly convex as 2 > 0.

- Function (b) → Using again the chain rule, we have that $h''(x) = 80(2x-3)^3$. Therefore, this function is neither concave nor convex (its second derivative can be both positive and negative, it depends on where you evaluate it).

- (Function (e) → $f''(x) = \frac{1}{x^2} < 0$ so this is strictly concave.

Exercise 3

Find the stationary points of the following functions. Are they max or min?

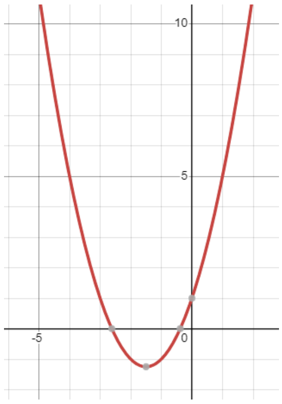
a) $f(x) = x^2 + 3x + 1$

Solution: This is the same equation as exercise 1(a). We know that $f'(x) = 2x + 3$, so its only stationary point is:

$$2x + 3 = 0$$

$$x = -\frac{3}{2}$$

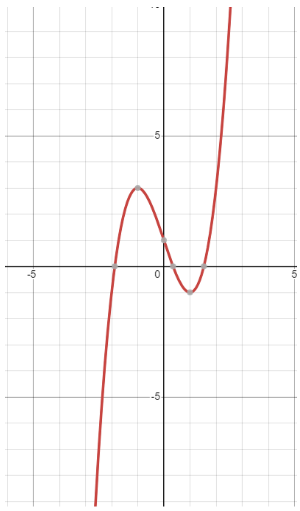
We now calculate the second derivative $f''(x) = 2$ The second derivative is positive (> 0) and $x = -\frac{3}{2}$ is a minimum as the function is convex. Another way to see this is that $f''(x)$ evaluated at $x = -\frac{3}{2}$ is 2.



b) $f(x) = x^3 - 3x + 1$

Solution: In this case $f'(x) = 3x^2 - 3$ and $f''(x) = 6$ so this function is neither concave nor convex. The first order condition is $f'(x) = 0$, which in this case yields two points: 1 and -1. To establish whether they are local max or min, we need to evaluate the second derivative at these points (second order condition). If we consider 1, then the second derivative is $f''(1) = 6$ >0 and x = 1 is a min.

If we consider -1, then the second derivative is and $f''(-1) = -6$ < 0 and X = -1 is a max.



Exercise 4

Consider the following matrices:

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 3 \\ 1 & 1 \\ 3 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

Find:

a) D = AB

Solution: Since A is 2x3 and B is 3x2, they can be multiplied and the resulting matrix D is going to be 2x2:

$$D = \begin{bmatrix} 5 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 5*4+1*1+0*0 & 5*3+1*1+0*2 \\ 2*4+1*1-1*0 & 2*3+1*1-1*2 \end{bmatrix} = \begin{bmatrix} 21 & 16 \\ 9 & 5 \end{bmatrix}$$

b) E = D + C

Solution: Since both matrices are 2x2 and B is 3x2, they can be added:

$$E = D + C = \begin{bmatrix} 21 & 16 \\ 9 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 21+2 & 16+1 \\ 9+2 & 5+3 \end{bmatrix} = \begin{bmatrix} 23 & 17 \\ 11 & 8 \end{bmatrix}$$

c) F=AC

Solution: Since A is a 2x3 matrix and C is a 2x2 matrix, they cannot be multiplied as the number of columns of A is not the same as the number of rows of C.

d) G = B'C

Solution: B' is the transpose of matrix B. In the transpose matrix we invert rows and columns. Therefore, B' is a 2x3 matrix and C is a 2x2 matrix. They cannot be multiplied as the number of columns of B' is not the same as the number of rows of C.

Exercise 5

Solve the following pair of equations:

$$-y = -8x - 4$$

$$y = 20x + 2$$

This is a system of 2 equations with 2 unknowns. We can rewrite it as:

$$\begin{aligned} 8x - y &= -4 \\ -20x + y &= 2 \end{aligned}$$

Which in matrix form can be written as $Ax = b$, where

$$A = \begin{bmatrix} 8 & -1 \\ -20 & 1 \end{bmatrix} \qquad x = \begin{bmatrix} x \\ y \end{bmatrix} \qquad b = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

We can solve it as $x = A^{-1}b$, but to invert the matrix A we need its determinant:

$$|A| = 8 * 1 - (-1) * (-20) = -12$$

So the inverse of A is:

$$A^{-1} = -\frac{1}{12} \begin{bmatrix} 1 & 1 \\ 20 & 8 \end{bmatrix} = \begin{bmatrix} -\frac{1}{12} & -\frac{1}{12} \\ -\frac{20}{12} & -\frac{8}{12} \end{bmatrix}$$

And finally we have:

$$x = A^{-1}b = \begin{bmatrix} -\frac{1}{12} & -\frac{1}{12} \\ -\frac{20}{12} & -\frac{8}{12} \end{bmatrix} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{12} \\ \frac{64}{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ \frac{16}{3} \end{bmatrix}$$

so $x = \frac{1}{6}$ and $y = \frac{16}{3}$