5.4 Kinetic energy of wind

The energy of wind is in the form of kinetic energy. For a wind speed u and air density ρ, the energy density E (i.e. energy per unit volume) is given by

$$E = \tfrac{1}{2}\rho u^2. \tag{5.1}$$

The volume of air flowing per second through a cross-sectional area A normal to the direction of the wind is uA. Hence the kinetic energy of the volume of air flowing per second through this area is given by $P = EuA$ or

$$P = \tfrac{1}{2}A\rho u^3. \tag{5.2}$$

Thus the power in the wind P varies as the cube of the wind speed u. Hence much more power is available at higher speeds and fluctuations in wind speed can cause the output of a wind turbine to vary significantly.

EXAMPLE 5.1

Calculate the power in a wind moving with speed $u = 5\ \mathrm{m\ s^{-1}}$ incident on a wind turbine with blades of 100 m diameter. How does the power change if the wind speed increases to $u = 10\ \mathrm{m\ s^{-1}}$? (Assume the density of air is $1.2\ \mathrm{kg\ m^{-3}}$.)

Substituting in eqn (5.2) we have

$$P = \tfrac{1}{2}A\rho u^3 = \tfrac{1}{2} \times (\pi \times 50^2) \times 1.2 \times 5^3 \approx 0.6\ \mathrm{MW}.$$

A power of 0.6 MW is sufficient to meet the average electricity usage of about 1000 European households. Doubling the wind speed increases the power by a factor of $2^3 = 8$, so the power would increase to $8 \times 0.6 = 4.8$ MW.

5.5 Principles of a horizontal-axis wind turbine

Unfortunately, not all of the power in the wind can be extracted by a wind turbine. This is because some of the kinetic energy is carried downstream of the turbine in order to maintain air flow. This effect places a theoretical maximum efficiency of 59% for extracting power from the wind, known as the Betz limit, which is described in detail in Derivation 5.1.

As the wind flows through a turbine it slows down as part of its energy is transferred to the turbine. The airflow looks like that shown in Fig. 5.5. Upstream the speed of the wind is u_0 and it passes through an area A_0. By the time the wind reaches the turbine it has slowed to u_1 and the area of the stream-tube has increased to A_1, the area swept out by the blades of the turbine. Downstream of the turbine the wind's cross-sectional area is A_2 and its speed is u_2. The drop in speed of the wind before and after the turbine gives rise to a pressure drop across the turbine, through Bernoulli's theorem, so there is a thrust on the turbine blades.

The maximum power is generated when downstream of the turbine the wind speed is one-third of the upstream speed u_0 and at the turbine the wind speed is two-thirds of u_0, i.e. $u_2 = \tfrac{1}{3}u_0$ and $u_1 = \tfrac{2}{3}u_0$ (see Derivation 5.1). Under these conditions the power extracted,

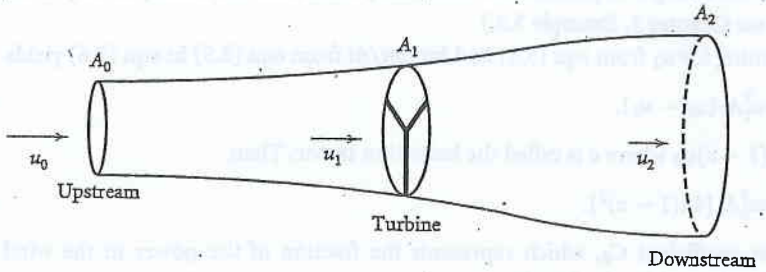

Fig. 5.5 Wind flow through a turbine.

P, is given by

$$P = \tfrac{1}{2}\rho A_1(16/27)u_0^3. \tag{5.3}$$

The power P_w in the wind passing through an area A_1 with a speed u_0 is given by eqn (5.2) as

$$P_w = \tfrac{1}{2}\rho A_1 u_0^3.$$

Thus, the fraction of the power extracted by the turbine, which is called the power coefficient C_P, given by

$$C_P = P/\left\{\tfrac{1}{2}\rho u_0^3 A_1\right\} \quad \text{or} \quad P = \tfrac{1}{2}C_P \rho u_0^3 A_1, \tag{5.4}$$

is $16/27 \simeq 59\%$ of the power in the incident wind passing freely through an area equal to that of the turbine, A_1. This limit for the power coefficient C_P of 16/27 of the incident wind power is called the Betz or Lanchester–Betz limit; it was first derived by Lanchester in 1915 and independently by Betz in 1921.

Derivation 5.1 Maximum extraction efficiency

We can obtain an estimate of the maximum efficiency by modelling the turbine as a thin disc (called an actuator disc) that extracts energy. Consider a stream-tube of air (see Chapter 3) shown in Fig. 5.5 that moves with speed u_1 through a wind turbine of cross-sectional area A_1. Upstream of the turbine the cross-sectional area of the stream-tube is A_0 and the air speed is u_0. Downstream of the turbine the cross-sectional area of the stream-tube is A_2 and the air speed is u_2.

Since the turbine extracts energy from the wind, the air speed decreases as it passes through the turbine and the cross-sectional area of the stream tube increases, as shown in Fig. 5.5. The thrust T exerted on the turbine by the wind is equal to the rate of change of momentum, so that

$$T = \frac{dm}{dt}(u_0 - u_2) \tag{5.5}$$

where dm/dt is the mass of wind flowing through the stream-tube per second.

The power P extracted is given by the product of the thrust and the air speed at the turbine, u_1, so that

$$P = Tu_1 = \frac{dm}{dt}(u_0 - u_2)u_1. \tag{5.6}$$

We can also express the power extracted as the rate of loss of kinetic energy of the wind, i.e.

$$P = \tfrac{1}{2}\frac{dm}{dt}\left(u_0^2 - u_2^2\right). \tag{5.7}$$

Comparing eqns (5.6) and (5.7), we require

$$(u_0 - u_2)u_1 = \tfrac{1}{2}\left(u_0^2 - u_2^2\right) = \tfrac{1}{2}(u_0 - u_2)(u_0 + u_2).$$

$$u_1 = \tfrac{1}{2}(u_0 + u_2), \text{ or}$$
$$u_2 = 2u_1 - u_0. \tag{5.8}$$

Also, by mass continuity (Chapter 3), the mass flow per second, dm/dt, is given by

$$\frac{dm}{dt} = \rho uA = \rho u_1 A_1. \tag{5.9}$$

(Note that the changes in pressure are sufficiently small that the density of air ρ is essentially constant; see Chapter 3, Example 3.3.)

Substituting for u_2 from eqn (5.8) and for dm/dt from eqn (5.9) in eqn (5.6) yields

$$P = 2\rho u_1^2 A_1(u_0 - u_1). \tag{5.10}$$

Let $u_1 = (1 - a)u_0$ where a is called the induction factor. Then

$$P = \tfrac{1}{2}\rho u_0^3 A_1\{4a(1 - a)^2\}. \tag{5.11}$$

The power coefficient C_P, which represents the fraction of the power in the wind that is extracted by the turbine, is given by

$$C_P = P/\left\{\tfrac{1}{2}\rho u_0^3 A_1\right\} = 4a(1 - a)^2. \tag{5.12}$$

Maximizing P, by setting dC_P/dt to zero, gives the maximum power extracted P_{max} when $a = \frac{1}{3}$ equal to

$$P_{max} = \tfrac{1}{2}\rho u_0^3 A_1\{16/27\}, \tag{5.13}$$

which is ~59% of the power in the incident wind passing freely through an area equal to that of the turbine, A_1. This limit for the power coefficient C_P of 16/27 of the incident wind power is called the Betz or Lanchester–Betz limit.