You may refer without proof to results from the course (theorems, examples, etc.).

Q1 Let $(\Omega, \mathcal{F}, \mu)$ be a measurable space with finite measure μ. The measure μ is called *non-atomic* if for every $A \in \mathcal{F}$ with $\mu(A) > 0$ there exists $B \in \mathcal{F}$ such that $B \subset A$ and $0 < \mu(B) < \mu(A)$. This question aims to show that the non-atomic measure assumes all intermediate values.

(i) For non-atomic finite measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, consider the function $F(x) = \mu((-\infty, x])$. Show that for every $0 < t < \mu(\mathbb{R})$ there exists $x \in \mathbb{R}$ such that $F(x) = t$.

(ii) Let μ be a non-atomic finite measure μ on $(\Omega, \mathcal{F})$. Show that for every $\varepsilon > 0$ there exists a finite collection of pairwise disjoint $\mathcal{F}$-measurable sets $A_1, \ldots, A_n$ (for some $n \geq 1$) such that $\bigcup_{k=1}^{n} A_k = \Omega$ and $\mu(A_k) \leq \varepsilon$ for $1, \ldots, n$. You may work in steps:

(1) Show that every set $A \in \mathcal{F}$ of positive measure has a measurable subset $B \subset A$ with $0 < \mu(B) \leq \varepsilon$.

(2) If $\mu(\Omega) > \varepsilon$ find A_1 such that $0 < \mu(A_1) \leq \varphi(\Omega)$, and then construct by induction a sequence (finite or infinite) of disjoint $\mathcal{F}$-measurable sets $A_1, A_2, \ldots$ satisfying

$$\frac{1}{2}\varphi\left(\Omega \setminus \bigcup_{k=1}^{m} A_k\right) \leq \mu(A_{m+1}) \leq \varepsilon,$$

where $\varphi(A) := \sup\{\mu(B) : B \in \mathcal{F}, B \subset A, \mu(B) \leq \varepsilon\}$ for $A \in \mathcal{F}$. If the sequence is finite, say $A_1, \ldots, A_n$, it should not be extendible by adding another set A_{n+1}.

(3) For $A_0 = \Omega \setminus \bigcup_k A_k$, where the union (finite or infinite) is over $k \geq 1$, show that $\mu(A_0) = 0$.

(iii) Use the result in part (ii) to prove that for $0 < t < \mu(\Omega)$ there exists $A \in \mathcal{F}$ with $\mu(A) = t$. Is it sufficient to only consider the case $t = \mu(\Omega)/2$?

Q2 A box has initially r red and g green balls. A ball is drawn at random, its colour is observed, and the ball is returned to the box together with d balls of this very colour. This is repeated many times. Let R_n be the number of red balls in the box after n draws.

(i) Show that the proportion of red balls

$$X_n = \frac{R_n}{nd + r + g}, \quad n \in \mathbb{N},$$

is a martingale relative to the natural filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

(ii) Argue that X_n converges almost surely and in the mean.

(iii) For $r = g = d = 1$ show by induction that the distribution of R_n is uniform on $\{1, \ldots, n+1\}$, and so find the limit distribution of X_n.

(iv) For $r = g = d = 1$ show that $(R_n - 1)/n,\ n \in \mathbb{N}$, is a *reversed* martingale (that is, the sequence has the martingale property if viewed backwards). What is the appropriate filtration in this case? You may use without proof that the reversed sequence is a Markov chain. Hint: compute the backward transition probabilities $\mathbb{P}\left[R_n = k | R_{n+1} = m\right]$ for $k = m - 1$ and $k = m$.

Q3 Let $(B(t),\ t \geq 0)$ be the Brownian motion.

(i) Determine

$$\mathbb{E}[B(t)|B(s) = x, B(u) = y]$$

for $s < t < u$ and all other possible orderings of positive distinct s, t, u (e.g. $t < s < u$).

(ii) Let $\tau_x := \inf\{t : B(t) = x\}$.

(a) Find explicitly the density of the random variable τ_x for $x > 0$ and show that $\mathbb{E}[\tau_x] = \infty$. Hint: recall the distribution of the maximum of the BM on a given interval $[0, t]$.

(b) Prove that the random variable $\tau_{x+y} - \tau_x$ is independent of τ_x and has the same distribution as τ_y for $x, y > 0$.

(c) Generalising (b), show that the random process $(\tau_x,\ x \geq 0)$ (having x in the role of time variable) has independent increments. That is, for $0 < x_1 < \cdots < x_n$ the random variables $\tau_{x_k} - \tau_{x_{k-1}}$ (where $\tau_0 = 0, k \leq n$) are independent.

(iii) Let $(W(t), t \geq 0)$ be another Brownian motion, independent of $(B(t), t \geq 0)$. For τ_x introduced in part (ii), find the probability density function of the random variable $W(\tau_x)$.

The two-dimensional process $((B(t), W(t)),\ t \geq 0)$ is called the planar Brownian motion. Then, τ_x is the time the process needs to first-hit a vertical line, and $W(\tau_x)$ is the position on this line.