Queen Mary
University of London

**Main Examination period 2024 – January – Semester A**

# MTH6107: Chaos & Fractals SOLUTIONS

**Examiners: O. Jenkinson, R. Klages**

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**You should attempt ALL questions. Marks available are shown next to the questions.**

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**Question 1 [24 marks].**

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 - 3x^2 + 3x$.

(i) Determine all fixed points of $f$. **[3]**

(ii) Determine, with justification, whether each fixed point is attracting or repelling. **[3]**

(iii) Determine the basin of attraction of each attracting fixed point. **[3]**

(iv) Give an example of an eventually periodic orbit that is not periodic, or explain why such points do not exist. **[3]**

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 1$.

(i) Determine, with justification, whether $f$ is topologically conjugate to $g_1 : \mathbb{R} \to \mathbb{R}$ defined by $g_1(x) = x + 2$. **[3]**

(ii) Determine, with justification, whether $f$ is topologically conjugate to $g_2 : \mathbb{R} \to \mathbb{R}$ defined by $g_2(x) = x - 1$. **[3]**

(iii) Determine, with justification, whether $f$ is topologically conjugate to $g_3 : \mathbb{R} \to \mathbb{R}$ defined by $g_3(x) = -x + 1$. **[3]**

(iv) Determine, with justification, whether $f$ is topologically conjugate to $g_4 : \mathbb{R} \to \mathbb{R}$ defined by $g_4(x) = x^2 - 1$. **[3]**

1(a)(i) Fixed points are 0, 1 and 2, since $f(x) - x = x(x^2 - 3x + 2) = x(x-1)(x-2)$.

(ii) Points 0 and 2 are repelling, and 1 is attracting.
Justification: $f'(x) = 3x^2 - 6x + 3$, so $|f'(0)| = 3 > 1$, $|f'(1)| = 0$, and $|f'(2)| = 3 > 1$, and a result from the module states that for a fixed point $p$, if $|f'(p)| < 1$ then $p$ is attracting, and if $|f'(p)| > 1$ then $p$ is repelling.

(iii) The basin of attraction of the fixed point 1 is $(0, 2)$ (note that if $x > 2$ or $x < 0$ then $|f^n(x)| \to \infty$ as $n \to \infty$).

(iv) There are no such points, since $f$ is injective, and a result from the module states that for injective maps every eventually periodic point is actually periodic. Injectivity follows from the fact that $f$ is strictly increasing, since $f'(x) = 3(x-1)^2 \geq 0$ for all $x \in \mathbb{R}$, and $f'(x) > 0$ for all $x \in \mathbb{R} \setminus \{1\}$.

(b)(i) Yes, $f$ and $g_1$ are conjugate: the map $h(x) = 2x$ is a homeomorphism, and $hf(x) = 2(x+1) = 2x + 2 = h(x) + 2 = g_1h(x)$.

(ii) Yes, $f$ and $g_2$ are conjugate. The map $h(x) = -x$ is a homeomorphism, and $hf(x) = -(x+1) = -x - 1 = h(x) - 1 = g_2h(x)$.

(iii) No, they are not conjugate: $f$ has no fixed point, whereas $g_3$ has a fixed point (at $x = 1/2$), and maps with a different number of fixed points cannot be topologically conjugate.

(iv) No, they are not conjugate: $f$ has no fixed points or period-2 points, whereas $g_4$ has two fixed points, and a 2-cycle $\{0, -1\}$.

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**Question 2 [27 marks].** For parameters $\lambda \in [0,1]$, define $f_\lambda : [0,1] \to [0,1]$ by $f_\lambda(x) = \lambda \sin(\pi x)$.

(a) Sketch the graphs of the functions $f_{1/4}$ and $f_1$. **[2]**

(b) Determine the value $\lambda_1 \in (0,1)$ such that the fixed point 0 is attracting for $\lambda \in [0, \lambda_1)$ and repelling for $\lambda \in (\lambda_1, 1]$. **[3]**

(c) Show that if $\lambda \in (\lambda_1, 1]$ then $f_\lambda$ has a non-zero fixed point. **[4]**

Henceforth, assume that for $\lambda \in (\lambda_1, 1]$ the non-zero fixed point of $f_\lambda$ is unique, and denoted by $x_\lambda$.

(d) Determine the value of $\lambda$ such that $x_\lambda = 1/6$. **[2]**

(e) Determine the value of $\lambda$ such that $x_\lambda = 1/2$. **[2]**

(f) Show that if $\lambda = 4\sqrt{3}/9$ then $x_\lambda = 2/3$. **[2]**

(g) Show that the point 1/6 is eventually periodic for the map $f_1(x) = \sin(\pi x)$. **[2]**

(h) Sketch the graph of $f_1^3$, taking care to mark the value of this function at the points $\alpha$, $\beta$, $1-\alpha$, $1-\beta$, where $\alpha = \frac{1}{\pi}\arcsin(1/6)$, $\beta = \frac{1}{\pi}\arcsin(5/6)$. **[3]**

(i) Show that $f_1$ has a point of least period 3. **[4]**

(j) Determine, with justification, whether $f_1$ has a point of least period 314159. **[3]**

2(a) One mark for each graph.
(b) $\lambda_1 = 1/\pi$. This is because $f'_\lambda(x) = \pi\lambda\cos(\pi x)$, so $|f'_\lambda(0)| = \pi\lambda$, which is strictly less than 1 if $\lambda < 1/\pi$, and strictly greater than 1 if $\lambda > 1/\pi$.

(c) Let $g_\lambda(x) = f_\lambda(x) - x$. Then $g'_\lambda(x) = \pi\lambda\cos(\pi x) - 1$. So $g_\lambda(0) = 0$ for all $\lambda$, and if $\lambda > \lambda_1$ then $g'_\lambda(0) = \pi\lambda - 1 > 0$ so $g'_\lambda(x) > 0$ for $x$ close enough to 0 (since $g_\lambda$ is $C^1$). Picking a particular such $x$ with $g'_\lambda(x) > 0$, we then note that $g_\lambda(1) < 0$, so by the Intermediate Value Theorem there exists $x_\lambda \in (x, 1)$ such that $g_\lambda(x_\lambda) = 0$, i.e. such that $f_\lambda(x_\lambda) = x_\lambda$, as required.

(d) The fixed point equation $x_\lambda = f_\lambda(x_\lambda)$ becomes $1/6 = \lambda\sin(\pi/6) = \lambda/2$, so $\lambda = 1/3$.

(e) $\lambda = 1/2$, since the fixed point equation $x_\lambda = f_\lambda(x_\lambda)$ becomes $1/2 = \lambda\sin(\pi/2) = \lambda$.

(f) If $\lambda = 4\sqrt{3}/9$ then $\lambda\sin(\pi(2/3)) = \lambda\sqrt{3}/2 = 4\sqrt{3}/9 \times \sqrt{3}/2 = 12/18 = 2/3$, so 2/3 is the non-zero fixed point.

(g) Now $f_1(1/6) = \sin(\pi/6) = 1/2$, so $f_1^2(1/6) = f_1(1/2) = \sin(\pi/2) = 1$, so $f_1^3(1/6) = f_1(1) = \sin(\pi) = 0$, and 0 is a fixed point. So 1/6 is an eventually fixed point, so in particular an eventually periodic point, for the map $f_1$.

(h) Graph has 8 humps, taking value 0 at the five points $0 < 1/6 < 1/2 < 5/6 < 1$, and in the 4 intervals between these points it takes the value 1 at $\alpha < \beta < 1-\beta < 1-\alpha$.

(i) The line $y = x$ intersects the graph of $f_1^3$ at 8 points. Two of these points are the fixed points for $f_1$, namely 0 and $x_\lambda$, and the other six points all have least period 3 under $f_1$.

(j) $f_1$ **does** have a point of least period 314159. Justification: Since $f_1$ is continuous, and from part (i) it has points of least period 3, Sharkovskii's Theorem implies $f_1$ has points of least period $n$ for all natural numbers $n$.

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**Question 3 [25 marks].** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 5/6 & \text{if } x < 0 \\ x + 5/6 & \text{if } 0 \leq x < 1/6 \\ 4/3 - 2x & \text{if } 1/6 \leq x < 2/3 \\ x - 2/3 & \text{if } 2/3 \leq x \leq 1 \\ 1/3 & \text{if } x > 1\,. \end{cases}$$

(a) Sketch the graph of $f$. **[3]**

(b) Determine the fixed point $p$ of $f$. **[3]**

(c) Determine the orbit under $f$ of the point 0. **[3]**

(d) Show that if $x \in (1/3, 2/3)$, with $x \neq p$, then there exists $N \in \mathbb{N}$ such that $f^N(x) \notin (1/3, 2/3)$. **[6]**

(e) Show that if $x \in [0, 1/3]$ then $f(x) \in [2/3, 1]$, and that if $x \in [2/3, 1]$ then $f(x) \in [0, 1/3]$. Use this to deduce that, for all integers $n \geq 0$, if $x \in [0, 1/3]$ then $f^{2n+1}(x) \in [2/3, 1]$, and that if $x \in [2/3, 1]$ then $f^{2n+1}(x) \in [0, 1/3]$. **[5]**

(f) Using (c), (d) and (e), or otherwise, determine the set of $n \in \mathbb{N}$ such that $f$ has an $n$-cycle. **[5]**

3(a) Graph is continuous, piecewise-linear, with maximum value 1 at 1/6, minimum value 0 at 2/3, and no other turning points.

(b) The fixed point is $p = 4/9$, i.e. the solution to $4/3 - 2x = x$.

(c) The orbit is $\{0, \frac{5}{6}, \frac{1}{6}, 1, \frac{1}{3}, \frac{2}{3}\}$, since
$0 \mapsto 5/6 \mapsto 5/6 - 2/3 = 1/6 \mapsto 1 \mapsto 1/3 \mapsto 4/3 - 2/3 = 2/3 \mapsto 0$.

(d) If $x \in (1/3, 2/3)$, with $x \neq p = 4/9$, then either $x \in (1/3, 4/9)$ or $x \in (4/9, 2/3)$. If $x \in (1/3, 4/9)$ then $f(x) > 4/9$, and the distance between $f(x)$ and $p$ is twice the distance between $x$ and $p$, since
$|f(x) - p| = f(x) - 4/9 = 8/9 - 2x = 2(4/9 - x) = 2|x - p|$. Similarly, If $x \in (4/9, 2/3)$ then $f(x) < 4/9$, and the distance between $f(x)$ and $p$ is twice the distance between $x$ and $p$, since $|f(x) - p| = 4/9 - f(x) = 2x - 8/9 = 2(x - 4/9) = 2|x - p|$. It follows that if $f^i(x) \in (1/3, 2/3)$ for $0 \leq i \leq n - 1$ then the distance between $f^n(x)$ and $p$ is $2^n$ times the distance between $x$ and $p$, and for sufficiently large $n$ the distance between $f^n(x)$ and $p$ will be larger than 1/3, so $f^n(x)$ will be outside of $(1/3, 2/3)$.

(e) If $x \in [0, 1/3]$ then either $x \in [0, 1/6)$ or $x \in [1/6, 1/3]$.
If $x \in [0, 1/6)$ then $f(x) = x + 5/6 \in [5/6, 1] \subset [2/3, 1]$. If $x \in [1/6, 1/3]$ then $x \geq 1/6$ so $f(x) = 4/3 - 2x \leq 4/3 - 1/3 = 1$, and $x \leq 1/3$ so $f(x) = 4/3 - 2x \geq 4/3 - 2/3 = 2/3$, so $f(x) \in [2/3, 1]$.
If $x \in [2/3, 1]$ then $f(x) = x - 2/3 \in [0, 1/3]$.

It follows that if $x \in [0, 1/3]$ then $f^n(x) \in [0, 1/3]$ for even $n$, and $f^n(x) \in [2/3, 1]$ for odd $n$, as required. Similarly, if $x \in [2/3, 1]$ then $f^n(x) \in [2/3, 1]$ for even $n$, and $f^n(x) \in [0, 1/3]$ for odd $n$, as required.

(f) The set is $\{1\} \cup \{2m : m \in \mathbb{N}\}$, i.e. the map $f$ has an $n$-cycle for $n = 1$ and all even natural numbers $n$, but not for any odd natural numbers $n > 1$.
Justification: Part (d) implies that the only periodic point in $(1/3, 2/3)$ is the fixed point $p$. Part (e) implies that if $x \in [0, 1/3] \cup [2/3, 1]$ then $x$ is not of period $n$ for any odd number $n$. By part (c) there is a 6-cycle, and since $f$ is continuous, Sharkovskii's Theorem then implies that $f$ has an $n$-cycle for all even $n$ (since 6 is larger than all other even numbers in the Sharkovskii order, but is smaller than all odd numbers except for 1).

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**Question 4 [24 marks].**

(a) Given an iterated function system in $\mathbb{R}^2$ defined by the 4 maps $\phi_0(x,y) = (x/3, y/3)$, $\phi_1(x,y) = ((x+2)/3, y/3)$, $\phi_2(x,y) = (x/3, (y+2)/3)$, $\phi_3(x,y) = ((x+2)/3, (y+2)/3)$, define $\Phi(A) = \cup_{i=0}^{3} \phi_i(A)$ for all sets $A \subset \mathbb{R}^2$, and let $F_k$ denote $\Phi^k([0,1]^2)$ for $k \geq 0$.

(i) Determine the set $F_1$. **[3]**

(ii) If $F_k$ is expressed as a disjoint union of $N_k$ closed squares, compute the number $N_k$. **[3]**

(iii) What is the common side length of each of the $N_k$ squares whose disjoint union equals $F_k$? **[3]**

(iv) Compute the box dimension of $F = \cap_{k=0}^{\infty} F_k$, being careful to justify your answer. **[5]**

(b) If $C \subset [0,1]$ denotes the middle third Cantor set, compute the box dimension of the set $C \times [0,1] = \{(x,y) : x \in C, y \in [0,1]\} \subset \mathbb{R}^2$. **[5]**

(c) (i) For a map $f : [0,1] \to \mathbb{R}$, how is the **set of non-escaping points** defined? **[2]**

(ii) Give an example, with justification, of a map $f$ whose set of non-escaping points has box dimension strictly smaller than 1/2. **[3]**

4(a)(i) $F_1$ consists of the 4 squares $[0,1/3] \times [0,1/3]$, $[0,1/3] \times [2/3,1]$, $[2/3,1] \times [0,1/3]$ and $[2/3,1] \times [2/3,1]$.

(ii) $F_k = 4^k$.

(iii) The side length is $(1/3)^k$.

(iv) If $\varepsilon_k = 1/3^k$ then $N(\varepsilon_k) = 4^k$, so the box dimension equals

$$\lim_{k\to\infty} \frac{\log N(\varepsilon_k)}{-\log \varepsilon_k} = \lim_{k\to\infty} \frac{\log 4^k}{-\log(1/3)^k} = \frac{\log 4}{\log 3}\,.$$

(b) The box dimension is $1 + \log 2/\log 3$.
One way of seeing this is to augment the iterated function system in part (a) with two additional maps, $\phi_4(x,y) = (x/3, (y+1)/3)$ and $\phi_5(x,y) = ((x+2)/3, (y+1)/3)$. The resulting iterated function system $\Psi$, given by $\Psi(A) = \cup_{i=0}^{5} \phi_i(A)$, is such that $\cap_{k=0}^{\infty} \Phi^k([0,1]^2) = C \times [0,1]$, and by a calculation analogous to the one in (a) we compute its box dimension to be

$$\lim_{k\to\infty} \frac{\log 6^k}{-\log(1/3)^k} = \frac{\log 6}{\log 3} = 1 + \frac{\log 2}{\log 3}\,.$$

(c)(i) The non-escaping set is $\{x \in [0,1] : f^n(x) \in [0,1] \text{ for all } n \geq 1\}$.

(ii) We might, for example, choose $f(x) = 8x \pmod 1$, as here the escaping set has box dimension equal to 1/3.
More generally, we could define $f(x) = mx \pmod 1$, for some suitably large natural number $m$. In this case the non-escaping set is equal to the set $\cap_{k=0}^{\infty}\Phi^k([0,1])$, where $\Phi(A) = \cup_{i=0}^{1}\phi_i(A)$, and $\phi_0(x) = x/m$, $\phi_1(x) = (x+m-1)/m$. The box dimension of this set is $\log 2/\log m$, and this is strictly smaller than 1/2 provided $m \geq 5$.

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**End of Paper.**

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