Main Examination period 2017

MTH6107 / MTH6107P: Chaos & Fractals SOLUTIONS

Duration: 2 hours

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Examiners: O. Jenkinson

Question 1. [22 marks]

(a) How is Sharkovsky's ordering of $\mathbb{N}$ defined? **[3]**

(b) State Sharkovsky's Theorem. **[3]**

(c) Let the map $f : \mathbb{R} \to \mathbb{R}$ be given by the formula $f(x) = 1 - ax^2$, where the constant $a \approx 1.75488$ is defined to be the real solution to the equation $a(1-a)^2 = 1$.

Show that the orbit under f of the point 1 is periodic. Determine the minimal period of this orbit, and whether this orbit is unstable, stable, or superstable. **[5]**

(d) Show that f has an orbit of minimal period n for every $n \in \mathbb{N}$. **[3]**

(e) Find all of the fixed points of f, and determine whether each fixed point is unstable, stable, or superstable. **[4]**

(f) Let F denote the restriction of f to the interval $[-1, 1]$ (i.e. $F : [-1, 1] \to [-1, 1]$ is defined by $F(x) = 1 - ax^2$).

(i) Is every periodic orbit for f also a periodic orbit for F? Justify your answer. **[2]**

(ii) Does F have an orbit of minimal period n for every $n \in \mathbb{N}$? Justify your answer. **[2]**

Solution:

(a) Sharkovsky's ordering $\prec$ of the natural numbers is given by:

$$\begin{gathered}
1 \prec 2 \prec 2^2 \prec 2^3 \prec \cdots \prec 2^m \prec \cdots \\
\vdots \\
\cdots \prec 2^k(2n-1) \prec \cdots \prec 2^k \cdot 7 \prec 2^k \cdot 5 \prec 2^k \cdot 3 \prec \cdots \\
\vdots \\
\cdots \prec 2(2n-1) \prec \cdots \prec 2 \cdot 7 \prec 2 \cdot 5 \prec 2 \cdot 3 \prec \cdots \\
\cdots \prec 2n-1 \prec \cdots \prec 7 \prec 5 \prec 3\,.
\end{gathered}$$

(b) Sharkovsky's Theorem says that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, and has a periodic orbit of minimal period n, then it has a periodic orbit of minimal period m for all $m \prec n$.

(c) Now $f(1) = 1 - a$, and $f(1-a) = 1 - a(1-a)^2 = 0$ by definition of a, and $f(0) = 1$, so the orbit $\{1, 1-a, 0\}$ is periodic under f, of minimal period 3.

The orbit is **superstable**, since the point 0 is in the orbit, and 0 is the critical point for f (thus $(f^3)'(1) = (f^3)'(0) = 0$).

(d) Since f is a continuous self-map of $\mathbb{R}$, and has an orbit of minimal period 3, then Sharkovsky's Theorem implies it has an orbit of minimal period n for every $n \in \mathbb{N}$, since 3 is the largest natural number in Sharkovsky's ordering.

(e) Fixed points are (real) solutions of $1 - ax^2 = x$, i.e. roots of the quadratic polynomial $ax^2 + x - 1$, and these are the two points x_+ and x_- given by

$$x_\pm = \frac{1}{2a}\left(-1 \pm \sqrt{1+4a}\right) .$$

Now $f'(x) = -2ax$, so $f'(x_+) = 1 - \sqrt{1+4a}$ and $f'(x_-) = 1 + \sqrt{1+4a}$. Since $a \approx 1.75488$ then $a > 7/4$ so $\sqrt{1+4a} > \sqrt{8}$. Therefore $f'(x_+) < 1 - \sqrt{8} < -1$, and $f'(x_-) > 1 + \sqrt{8} > 1$. It follows that both of the fixed points $x_\pm$ are **unstable**, since $|f'(x_\pm)| > 1$.

(f) (i) This is not the case: f has two fixed points, but F only has a single fixed point $x_+ = \frac{1}{2a}\left(-1 + \sqrt{1+4a}\right)$ (note that x_- is not a fixed point for F since $x_- = \frac{1}{2a}\left(-1 - \sqrt{1+4a}\right) < -1$ does not lie in $[-1,1]$).

(ii) F does have an orbit of minimal period n for every $n \in \mathbb{N}$. To see this it suffices to apply Sharkovsky's theorem to a continuous map $g : \mathbb{R} \to \mathbb{R}$ whose restriction to $[-1,1]$ equals F and which has no periodic points in $\mathbb{R} \setminus [-1,1]$. Now $F(1) = F(-1) = 1 - a$, so define $g : \mathbb{R} \to \mathbb{R}$ to equal the constant $1 - a$ on $\mathbb{R} \setminus [-1,1]$, and to equal F on $[-1,1]$; this g has the required properties, since the points in $\mathbb{R} \setminus [-1,1]$ are not periodic (in fact they are pre-periodic). Now g has an orbit of minimal period 3 (namely $\{1, 1-a, 0\}$), so by Sharkovsky it has an orbit of minimal period n for every $n \in \mathbb{N}$, therefore so does F.

Question 2. [30 marks] Let $\mathscr{H}$ denote the collection of compact subsets of $\mathbb{R}$. Let $\Phi : \mathscr{H} \to \mathscr{H}$ be the iterated function system defined by the two maps $\phi_1(x) = x/5$ and $\phi_2(x) = (x+4)/5$, and let C_k denote $\Phi^k([0,1])$ for $k \geqslant 0$.

(a) For $A, B \in \mathscr{H}$, what is the definition of the **Hausdorff distance** $h(A,B)$? **[5]**

(b) Write down the sets C_1 and C_2. **[4]**

(c) Compute $h(C_1, C_2)$. **[5]**

(d) If C_k is expressed as a disjoint union of N_k closed intervals, compute the number N_k. **[3]**

(e) What is the common length of each of the N_k closed intervals whose disjoint union equals C_k? **[3]**

(f) Given a set $A \subset \mathbb{R}$, what is the definition of its **box dimension**? **[5]**

(g) Using your answers to parts (d) and (e), or otherwise, show that if the box dimension of $C = \cap_{k=0}^{\infty} C_k$ exists then it must equal $\log 2 / \log 5$. **[5]**

Solution:

(a) Let $d(\cdot,\cdot)$ be the usual distance function on $\mathbb{R}$. For $A \in \mathscr{H}$, and $x \in \mathbb{R}$, define $\rho(x,A) = \min_{y \in A} d(x,y)$.

Then define $h_{BA} = \max_{x \in B} \rho(x,A)$, and finally set

$$h(A,B) = \max(h_{AB}, h_{BA}).$$

(b) $C_1 = [0, 1/5] \cup [4/5, 1]$, and

$$C_2 = [0, 1/25] \cup [4/25, 1/5] \cup [4/5, 21/25] \cup [24/25, 1].$$

(c) If $A = C_1$, $B = C_2$ then $h_{BA} = 0$ since $B \subset A$, whilst

$$h_{AB} = \max_{x \in C_1} \rho(x, C_2) = \rho(1/10, C_2) = \rho(1/10, 1/25) = 3/50,$$

so

$$h(C_1, C_2) = \max(3/50, 0) = 3/50.$$

(d) $N_k = 2^k$ because $N_0 = 1$ and the recursive procedure doubles the number of intervals at each step.

(e) The length is $1/5^k$, because the length of the closed intervals decreases by a factor of 5 at each step, and the length of $C_0 = [0,1]$ is 1.

(f) For $\varepsilon > 0$ let $N(\varepsilon)$ denote the smallest number of length-ε intervals needed to cover A. The box dimension of A is then

$$\lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{-\log \varepsilon},$$

provided the limit exists.

(g) If $\varepsilon_k = 1/5^k$ then $N(\varepsilon_k) = 2^k$, by parts (d) and (e), and so the box dimension equals

$$\lim_{k \to \infty} \frac{\log N(\varepsilon_k)}{-\log \varepsilon_k} = \lim_{k \to \infty} \frac{k \log 2}{k \log 5} = \frac{\log 2}{\log 5}.$$

Question 3. [22 marks]

(a) If X and Y are intervals in $\mathbb{R}$, explain what it means for two maps $f : X \to X$ and $g : Y \to Y$ to be **topologically conjugate**. **[3]**

(b) Show that if h is a topological conjugacy between f and g, then h is also a topological conjugacy between f^n and g^n, for all integers $n \geqslant 1$. **[5]**

(c) Using (b) above, or otherwise, show that if f and g are topologically conjugate then for each $n \in \mathbb{N}$, every period-n orbit for f is mapped by the conjugacy to a period-n orbit for g. **[5]**

(d) If $f : [0,\infty) \to [0,\infty)$ is defined by $f(x) = 4x$, and $g : [0,\infty) \to [0,\infty)$ is defined by $g(x) = 2x$, use the map $h(x) = \sqrt{x}$ to show that f and g are topologically conjugate. **[5]**

(e) Determine whether the map $F : [0,1) \to [0,1)$ given by $F(x) = 4x \pmod 1$ is topologically conjugate to the map $G : [0,1) \to [0,1)$ given by $G(x) = 2x \pmod 1$, being careful to justify your answer. **[4]**

Solution:

(a) f and g are topologically conjugate if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$.

(b) We claim that $h \circ f^n = g^n \circ h$ for all $n \geqslant 1$, asuuming that the case $n = 1$ holds. The proof is by induction. Let us make the inductive hypothesis that the equation holds for $n = k$. Then

$$h \circ f^{k+1} = h \circ f^k \circ f = g^k \circ h \circ f = g^k \circ g \circ h = g^{k+1} \circ h$$

where we have used the inductive hypothesis, and the case $n = 1$. So the equation holds for $n = k+1$, so the proof by induction is complete.

(c) Let $\{f^i(x) : 0 \leqslant i \leqslant n-1\}$ be a period-n orbit for f. The image under h is $\{h(f^i(x)) : 0 \leqslant i \leqslant n-1\}$, which we can write as $\{g^i(h(x)) : 0 \leqslant i \leqslant n-1\}$ by (b) above, so the image set consists of iterates under g of the point $h(x)$. The orbit is periodic (of period n) because $g^n(h(x)) = h(f^n(x)) = h(x)$ (by (b) above, and because x has period n under f), as required.

(d) The map $h(x) = \sqrt{x}$ is a homeomorphism of $[0,\infty)$. Now $h(f(x)) = \sqrt{f(x)} = \sqrt{4x} = 2\sqrt{x} = 2h(x) = g(h(x))$, so h is a topological conjugacy, as required.

(e) F and G are **not** topologically conjugate. One way of seeing this is to note that F only has a single fixed point (namely at 0), whereas G has 3 fixed points (at 0, $1/3$, and $2/3$). This contradicts (c) above, which tells us that if the maps are topologically conjugate then there is a one-to-one correspondence between fixed points for f and fixed points for g.

Question 4. [26 marks] Suppose $f : [0,1] \to [0,1]$.

(a) If f is C^1, what is the definition of the **Lyapunov exponent** $\lambda(x)$ of f at $x \in [0,1]$? **[3]**

(b) If f is C^1 and x is a point of minimal period N, what is the definition of its **multiplier**? **[2]**

(c) Use the Intermediate Value Theorem to show that if f is continuous then it has at least one fixed point. **[8]**

(d) Show that if f is continuous and order reversing (i.e. $f(x) > f(y)$ whenever $x, y \in [0,1]$ satisfy $x < y$) then f has a unique fixed point (you may use the result from part (c) above). **[5]**

(e) Does there exist a continuous map $g : (0,1) \to (0,1)$ which has no fixed points? Justify your answer. **[4]**

(f) Does there exist a discontinuous order reversing map $h : [0,1] \to [0,1]$ which has no fixed points? Justify your answer. **[4]**

Solution:

(a) The Lyapunov exponent is defined by

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log|f'(f^i x)|,$$

provided this limit exists.

(b) The multiplier is defined to equal $(f^N)'(x)$.

(c) Define $\phi : [0,1] \to \mathbb{R}$ by $\phi(x) = f(x) - x$, noting that ϕ is continuous because f is. Note that a point $c \in [0,1]$ is a fixed point for f if and only if $\phi(c) = 0$.

If $f(0) = 0$ or $f(1) = 1$ then f certainly has a fixed point, so suppose that $f(0) \neq 0$ and $f(1) \neq 1$.

Now $f(0) \in (0,1]$, so $\phi(0) = f(0) > 0$; similarly, $f(1) \in [0,1)$, so $\phi(1) = f(1) - 1 < 0$. Using these inequalities $\phi(1) < 0 < \phi(0)$, the intermediate value theorem implies that there exists $c \in (0,1)$ such that $\phi(c) = 0$. This value c is a fixed point for f, as required.

(d) Existence of a fixed point is guaranteed by (c) above. To prove uniqueness, suppose that there are two points $c, d \in [0,1]$, with $c < d$, which are both fixed points for f. Since f is order reversing, the inequality $c < d$ implies that $f(c) > f(d)$, but this means that $c > d$ (since both points are fixed points of f), contradicting the assumption that $c < d$. Therefore there can in fact be only one fixed point.

(e) Such maps **do** exist, and it suffices to give a single example, e.g. $g(x) = x/2$.

(f) Such maps **do** exist, and it suffices to give a single example, e.g. the map $h : [0,1] \to [0,1]$ defined by $h(x) = 1 - x/2$ for $x \in [0,1/2)$ and $h(x) = (1-x)/2$ for $x \in [1/2,1]$.

End of Paper.