Queen Mary
University of London

MTH6107 / MTH6107P: Chaos & Fractals (SOLUTIONS)

Duration: 2 hours

Date and time: Summer 2016

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Examiner(s): O. Jenkinson

Question 1. [27 marks]

(a) For a differentiable map $f : \mathbb{R} \to \mathbb{R}$, how is the **multiplier** of a periodic orbit defined? **[2]**

(b) Write down a condition on the multiplier which guarantees that a periodic orbit is **stable** (i.e. **attractive**). **[2]**

(c) Let $f_\lambda : [-1, 1] \to [-1, 1]$ be the logistic map, defined by $f_\lambda(x) = 1 - \lambda x^2$ for parameters $\lambda \in [0, 2]$.

(i) For $\lambda \in [0, 2)$, compute the fixed point $x^* = x^*(\lambda) \in [-1, 1]$ of f_λ. **[3]**

(ii) Compute the multiplier of this fixed point $x^*(\lambda)$. **[3]**

(iii) Determine the largest value λ_1 with the property that the fixed point $x^*(\lambda)$ is stable for all $\lambda \in [0, \lambda_1)$. **[2]**

(iv) For $\lambda > \lambda_1$, determine the periodic orbit of f_λ which has minimal period 2. **[6]**

(v) Compute the multiplier of this period-2 orbit, and determine the largest value λ_2 with the property that this orbit is stable for all $\lambda \in (\lambda_1, \lambda_2)$. **[4]**

(vi) Briefly define what is meant by a **period-doubling bifurcation**. **[2]**

(vii) How is the **Feigenbaum constant** δ defined? **[3]**

Solution:

(a) If the orbit is generated by the point x, of minimal period n, the multiplier is defined to be $(f^n)'(x)$. An alternative expression (courtesy of the chain rule) is $\prod_{i=0}^{n-1} f'(f^i x)$. **[2]**

(b) If the multiplier is strictly smaller than 1 in absolute value then the orbit is stable. **[2]**

(c) (i) Fixed points of f_λ satisfy $\lambda x^2 + x - 1 = 0$, so $x = \frac{-1 \pm \sqrt{1+4\lambda}}{2\lambda}$, of which only

$$x^*(\lambda) = \frac{-1 + \sqrt{1 + 4\lambda}}{2\lambda}$$

belongs to $[-1, 1]$ when $\lambda \in [0, 2)$. **[3]**

(ii) The multiplier is

$$f_\lambda'(x^*(\lambda)) = -2\lambda x^*(\lambda) = 1 - \sqrt{1 + 4\lambda}\,. \quad \textbf{[3]}$$

(iii) $\lambda_1 = 3/4$. This is because the multiplier is a strictly decreasing function of λ, decreasing from value 0 at $\lambda = 0$ to value -1 at $\lambda = 3/4$. **[2]**

(iv) The period-2 points satisfy $f_\lambda^2(x) - x = 0$. But

$$f_\lambda^2(x) = 1 - \lambda(1-\lambda x^2)^2 = -\lambda^3 x^4 + 2\lambda^2 x^2 - \lambda + 1\,,$$

so

$$f_\lambda^2(x) - x = -\lambda^3 x^4 + 2\lambda^2 x^2 - x - \lambda + 1\,.$$

But both **fixed** points are roots of this polynomial, so $\lambda x^2 + x - 1$ is a factor of this polynomial, hence we can factorise $f_\lambda^2(x) - x$ as

$$f_\lambda^2(x) - x = -(\lambda x^2 + x - 1)(\lambda^2 x^2 - \lambda x + (1-\lambda))\,.$$

Therefore the points of **minimal** period 2 are the roots of $\lambda^2 x^2 - \lambda x + (1-\lambda)$, namely

$$x_\pm(\lambda) = \frac{1 \pm \sqrt{4\lambda - 3}}{2\lambda}\,. \quad \textbf{[6]}$$

(v) The multiplier for this period-2 orbit is then the product of

$$f'_\lambda(x_+(\lambda)) = -2\lambda x_+(\lambda) = -(1 + \sqrt{4\lambda-3})$$

and

$$f'_\lambda(x_-(\lambda)) = -2\lambda x_-(\lambda) = -(1 - \sqrt{4\lambda-3})\,,$$

namely

$$1 - (4\lambda - 3) = 4 - 4\lambda = 4(1-\lambda)\,.$$

This multiplier decreases from value 1 at $\lambda = \lambda_1 = 3/4$ to value -1 at $\lambda = 5/4$. We therefore see that

$$\lambda_2 = 5/4\,. \quad \textbf{[4]}$$

(vi) A **period-doubling bifurcation** is the event such as occurs at $\lambda = \lambda_1$, or alternatively at $\lambda = \lambda_2$, whereby a formerly stable period-n orbit loses its stability, and a new stable period-$2n$ orbit is born. [**2**]

(vii) If we denote by (λ_n) the sequence of parameter values at which the period-doubling bifurcations occur, the Feigenbaum constant δ can be defined by:

$$\delta = \lim_{n\to\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n}\,. \quad \textbf{[3]}$$

Question 2. [26 marks]

(a) Given a subset of $\mathbb{R}^2$, how is its **box dimension** defined? **[4]**

(b) Briefly describe the construction of the **Sierpinski triangle** P^*. Use this description to show that if the box dimension of P^* exists then it must equal $\log 3/\log 2$. **[8]**

(c) Let $\mathcal{H}$ denote the collection of compact subsets of $\mathbb{R}^2$. For $A, B \in \mathcal{H}$, how is the **Hausdorff distance** $h(A, B)$ defined? **[4]**

(d) Given a finite collection of self-maps of $\mathbb{R}^2$, how is the corresponding **iterated function system** defined? **[4]**

(e) What does it mean for a self-map of $\mathbb{R}^2$ to be a **contraction mapping**? **[3]**

(f) State the Dubins & Freedman Theorem on iterated function systems consisting of contraction mappings. **[3]**

Solution:

(a) For $\varepsilon > 0$ let $N(\varepsilon)$ denote the smallest number of squares of side length ε needed to cover A. The box dimension of A is then

$$\lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{-\log \varepsilon},$$

provided the limit exists. **[4]**

(b) Begin with a solid equilateral triangle, then sub-divide it into 4 congruent equilateral triangles, then remove the central triangle, leaving 3 solid equilateral triangles.

Repeat the above step with each of the remaining 3 triangles, and continue the process ad infinitum. **[4]**

Assuming (without loss of generality) that the initial equilateral triangle has side length 1, we see that $N(1/2) = 3$, and more generally $N(1/2^k) = 3^k$, so existence of the box dimension D means that

$$D = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{-\log \varepsilon} = \lim_{k \to \infty} \frac{\log N(1/2^k)}{-\log 2^{-k}} = \lim_{k \to \infty} \frac{\log 3^k}{k \log 2} = \frac{\log 3}{\log 2}. \quad \textbf{[4]}$$

(c) Let $d(\cdot, \cdot)$ be the usual distance function on $\mathbb{R}^2$. For $A \in \mathcal{H}$, and $x \in \mathbb{R}^2$, define $\varrho(x, A) = \min_{y \in A} d(x, y)$.

Then define $h_{BA} = \max_{x \in B} \varrho(x, A)$, and finally set

$$h(A, B) = \max(h_{AB}, h_{BA}). \quad \textbf{[4]}$$

(d) If the self-maps of $\mathbb{R}^2$ are ϕ_i, for $i = 1, \ldots, n$, then the corresponding iterated function system is the self-map Φ of $\mathcal{H}$ defined by

$$\Phi(A) = \cup_{i=1}^{n} \phi_i(A)$$

for all $A \in \mathcal{H}$. **[4]**

(e) $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is a contraction mapping if there exists a constant $\alpha \in [0, 1)$ such that

$$d(\phi(z), \phi(w)) \leqslant \alpha d(z, w)$$

for all $w, z \in \mathbb{R}^2$, where d is the usual Euclidean distance. **[3]**

(f) The Dubins-Freedman theorem states that given contraction mappings $\phi_i : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1, \ldots, n$, the associated iterated function system $\Phi : \mathcal{H} \to \mathcal{H}$ has a unique fixed point. The fixed point is attracting, and its basin of attraction is the whole of $\mathcal{H}$. **[3]**

Question 3. [25 marks]
Let Σ denote the interval $[-1, 1]$.

(a) Explain what it means for two maps $f, g : \Sigma \to \Sigma$ to be **topologically conjugate**. **[3]**

(b) Show that the notion of topological conjugacy defines an equivalence relation on the set of self-maps of Σ. **[4]**

(c) Use the map $h(x) = \sin(\pi x/2)$ to show that the map $f : \Sigma \to \Sigma$ defined by $f(x) = 1 - 2|x|$ is topologically conjugate to the Ulam map $g : \Sigma \to \Sigma$ given by $g(x) = 1 - 2x^2$. **[6]**

(d) Find the fixed point of the map $G : \Sigma \to \Sigma$ defined by $G(x) = 1 - x^2$, and determine, with justification, whether this point is unstable, stable, or superstable. **[4]**

(e) Find the periodic orbit of minimal period 2 for G, and determine, with justification, whether this orbit is unstable, stable, or superstable. **[4]**

(f) Determine whether the map $F : \Sigma \to \Sigma$ given by $F(x) = 1 - |x|$ is topologically conjugate to G, being careful to justify your answer. **[4]**

Solution:

(a) f and g are topologically conjugate if there exists a homeomorphism $h : \Sigma \to \Sigma$ such that $h \circ f = g \circ h$. **[3]**

(b) Clearly any f is topologically conjugate to itself: just take h to be the identity map. **[1]**

The relation is symmetric: if $h \circ f = g \circ h$ then $H \circ g = f \circ H$ where $H = h^{-1}$. **[1]**

The relation is transitive: if $h \circ f_1 = f_2 \circ h$ and $h' \circ f_2 = f_3 \circ h'$, then setting $H = h' \circ h$ we see that

$$H \circ f_1 = h' \circ h \circ f_1 = h' \circ f_2 \circ h = f_3 \circ h' \circ h = f_3 \circ H\,.$$ **[2]**

(c) First observe that $h : \Sigma \to \Sigma$ defined by $h(x) = \sin(\pi x/2)$ is indeed a homeomorphism. **[1]**

We will show that $h \circ f = g \circ h$.

Firstly, if $x \in [-1, 0]$ then
$h(f(x)) = \sin((2x + 1)\pi/2) = \sin(\pi/2 + \pi x) = \cos(\pi x)$,

and if $x \in [0, 1]$ then
$h(f(x)) = \sin((1 - 2x)\pi/2) = \sin(\pi/2 - \pi x) = \cos(\pi x)$.

Secondly, $g(h(x)) = 1 - 2\sin^2(\pi x/2) = \cos \pi x$.

So $g(h(x)) = h(f(x))$, as required. **[5]**

(d) The fixed point x_* satisfies $x_* = 1 - x_*^2$, so equals $\frac{1}{2}(-1 \pm \sqrt{5})$. Now $\frac{1}{2}(-1 - \sqrt{5}) < -1$ so is outside Σ, therefore the required fixed point is $\frac{1}{2}(-1 + \sqrt{5})$. **[2]**

Now $G'(x) = -2x$, so $G'(x_*) = 1 - \sqrt{5} < -1$, so this fixed point is **unstable**. **[2]**

(e) The orbit of minimal period 2 is $\{0, 1\}$. **[2]**

Since $G'(0) = 0$ we see that this orbit is **superstable**. **[2]**

(f) The two maps are **not** topologically conjugate. **[2]**

Justification: Every point in $[0, 1]$ has minimal period 2 under F, whereas G only has a single orbit of minimal period 2, therefore the maps cannot be topologically conjugate. **[2]**

Question 4. [22 marks]
Let $\sigma : [0, 1) \to [0, 1)$ and $\tau : [0, 1) \to [0, 1)$ be defined by $\sigma(x) = 2x \pmod 1$ and $\tau(x) = 3x \pmod 1$.

(a) Given $x \in [0, 1)$, with binary expansion $x = \sum_{k=1}^{\infty} b_k/2^k$ where each $b_k \in \{0, 1\}$, show that x is periodic under σ if and only if the binary digit sequence $(b_k)_{k=1}^{\infty}$ is periodic. **[10]**

(b) Determine the period-5 orbit of σ which is contained in the interval $[3/20, 13/20]$. **[3]**

(c) Determine the periodic orbit of σ which is contained in the interval $[3/10, 4/5]$. **[3]**

(d) Identify, with justification, those points of minimal period 4 for σ which are also of minimal period 4 for τ. **[6]**

Solution:

(a) Applying the doubling map σ corresponds to a (left) shift of the binary digit sequence, so if

$$x = .b_1 b_2 \dots b_T b_1 b_2 \dots b_T \dots$$

is such that the digit sequence has period T, then $\sigma^T(x) = x$, so x is periodic under σ. **[3]**

Conversely, if x is periodic with period T, then $x = \sigma^T(x) = 2^T x \pmod 1$, so $x(2^T - 1) =: m \in \{1, 2, \dots, 2^T - 2\}$, therefore

$$x = \frac{m}{2^T - 1} = \frac{m}{2^T} \frac{1}{1 - 2^{-T}} = \frac{m}{2^T} \left(1 + 2^{-T} + 2^{-2T} + 2^{-3T} + \dots\right) . \quad \textbf{[3]}$$

Now let $b_1, \dots, b_T \in \{0, 1\}$ be such that

$$m = b_1 2^{T-1} + b_2 2^{T-2} + \dots + b_T 2^0$$

so

$$\frac{m}{2^T} = \frac{b_1}{2} + \frac{b_2}{2^2} + \dots + \frac{b_T}{2^T} , \qquad \textbf{[2]}$$

therefore

$$x = \left(\frac{b_1}{2} + \frac{b_2}{2^2} + \dots + \frac{b_T}{2^T}\right) \left(1 + 2^{-T} + 2^{-2T} + 2^{-3T} + \dots\right) ,$$

in other words

$$x = .b_1 b_2 \dots b_T b_1 b_2 \dots b_T \dots ,$$

so the digit sequence is periodic. **[2]**

(b) The unique such orbit is $\{5/31, 10/31, 20/31, 9/31, 18/31\}$. (Note: To arrive at this answer probably requires enumerating other period-5 orbits). **[3]**

(c) The unique such orbit is $\{1/3, 2/3\}$. (Note: To arrive at this answer probably requires enumerating other periodic orbits). [**3**]

(d) The points of period 4 for σ are those rationals of the form $m/15 = m/(2^4 - 1)$ for $m \in \{0, 1, \ldots, 14\}$, and all of these points except for 0, 1/3 and 2/3 have minimal period 4.

We deduce there are 3 orbits of minimal period 4, namely

$$\{1/15, 2/15, 4/15, 8/15\},$$

$$\{1/5, 2/5, 4/5, 3/5\},$$

and

$$\{7/15, 14/15, 13/15, 11/15\}.$$

Under τ, the orbit $\{1/5, 2/5, 4/5, 3/5\}$ has minimal period 4, because $\tau(1/5) = 3/5$, $\tau(3/5) = 4/5$, $\tau(4/5) = 2/5$, $\tau(2/5) = 1/5$.

Under τ the points in $\{1/15, 2/15, 4/15, 8/15\}$ or $\{7/15, 14/15, 13/15, 11/15\}$ are pre-periodic but not periodic.

Therefore the set of points of minimal period 4 for σ which are also of minimal period 4 for τ is precisely $\{1/5, 2/5, 4/5, 3/5\}$.

End of Paper.