Queen Mary
University of London

B. Sc. Examination by course unit 2015

MTH6107: Chaos & Fractals (SOLUTION SHEET)

Duration: 2 hours

Date and time: May 2015, 14.30–16.30

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You should attempt ALL questions. Marks awarded are shown next to the questions.

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Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Examiner(s): O. Jenkinson

Question 1. (a) [4 marks] For a map $f : \Sigma \to \Sigma$ on a non-empty set Σ, what does it mean to say that $x \in \Sigma$ is a *periodic point* for f, and how is its *minimal period* defined?

(b) [6 marks] Give a detailed statement of Sharkovsky's Theorem.

(c) [6 marks] Order the integers from 1 to 25 inclusive using Sharkovsky's ordering.

(d) [4 marks] For the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x-1)(1-3x^2/2)$, determine the orbit of the point 0.

(e) [4 marks] Show that the map f of part (d) above has a point of minimal period n for every $n \in \mathbb{N}$.

Solution:

(a) [4 marks] It means that $f^n(x) = x$ for some $n \in \mathbb{N}$. Its minimal period is the smallest natural number N such that $f^N(x) = x$.

(b) [6 marks] Sharkovsky's ordering $\prec$ of the natural numbers is given by:

$$
\begin{gathered}
1 \prec 2 \prec 2^2 \prec 2^3 \prec \cdots \prec 2^m \prec \cdots \\
\vdots \\
\cdots \prec 2^k(2n-1) \prec \cdots \prec 2^k \cdot 7 \prec 2^k \cdot 5 \prec 2^k \cdot 3 \prec \cdots \\
\vdots \\
\cdots \prec 2(2n-1) \prec \cdots \prec 2 \cdot 7 \prec 2 \cdot 5 \prec 2 \cdot 3 \prec \cdots \\
\cdots \prec 2n-1 \prec \cdots \prec 7 \prec 5 \prec 3\,.
\end{gathered}
$$

Sharkovsky's Theorem then says that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, and has a periodic orbit of minimal period n, then it has a periodic orbit of minimal period m for all $m \prec n$.

(c) [6 marks]

$1 \prec 2 \prec 4 \prec 8 \prec 16 \prec 24 \prec 20 \prec 12 \prec 22 \prec 18 \prec 14 \prec 10 \prec 6 \prec 25 \prec 23 \prec 21 \prec 19 \prec 17 \prec 15 \prec 13 \prec 11 \prec 9 \prec 7 \prec 5 \prec 3$

(d) [4 marks] It is the period-3 orbit $\{0, -1, 1\}$.

(e) [4 marks] The map f is certainly continuous, so the existence of an orbit of minimal period 3 implies, by Sharkovsky's Theorem, the existence of points of minimal period n for all $n \in \mathbb{N}$.

Question 2. Suppose the map $f : [0,1] \to [0,1]$ is defined by

$$f(x) = \begin{cases} 8x^3 & \text{if } x \in [0,1/2] \\ 2(1-x) & \text{if } x \in (1/2,1]\,. \end{cases}$$

(a) [6 marks] Determine the three fixed points of f.

(b) [6 marks] Compute the multiplier of each fixed point, and use this to determine whether the point is unstable, stable, or superstable.

(c) [5 marks] For $x = 2/5$, compute the points $f(x)$, $f^2(x)$, and $f^3(x)$.
Describe, with justification, the behaviour of $f^n(x)$ as $n \to \infty$.

Solution:

(a) [6 marks] There are 3 fixed points, at 0 and $1/\sqrt{8}$ (i.e. the two solutions to $x = 8x^3$ in $[0,1]$), and at $2/3$ (i.e. the solution to $x = 2(1-x)$).

(b) [6 marks] Since

$$f'(x) = \begin{cases} 24x^2 & \text{if } x \in [0,1/2) \\ -2 & \text{if } x \in (1/2,1] \end{cases}$$

we see that:

the multiplier at 0 is $f'(0) = 0$, a *superstable* fixed point;

the multiplier at $1/\sqrt{8}$ is $f'(1/\sqrt{8}) = 3$, an *unstable* fixed point;

the multiplier at $2/3$ is -2, an *unstable* fixed point.

(c) [5 marks] $f(x) = 8(2/5)^3 = 64/125$,

$f^2(x) = 2(1 - 64/125) = 122/125$,

$f^3(x) = 2(1 - 122/125) = 6/125$.

Now the point $f^3(x) = 6/125$ lies between 0 and $1/\sqrt{8}$, so is in the basin of attraction of the fixed point 0, so $f^n(x) \to 0$ as $n \to \infty$.

Question 3. (a) [9 marks] Define what it means for $f : \mathbb{R} \to \mathbb{R}$ to be

(i) a *homeomorphism*,

(ii) a *diffeomorphism*,

(iii) *order reversing.*

(b) [10 marks] Prove that an order reversing diffeomorphism $f : \mathbb{R} \to \mathbb{R}$ has exactly one fixed point.

Solution:

(a) (i) [3 marks] A homeomorphism is a continuous bijection whose inverse map is also continuous.

(ii) [3 marks] A diffeomorphism is defined (in this module) to be a bijection such that both f and f^{-1} are C^1 maps, i.e. they are differentiable with continuous derivative.

(iii) [3 marks] It means that if $x < y$ then $f(x) > f(y)$.

(b) [10 marks] Existence: The fact that f is an order reversing diffeomorphism means that $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = +\infty$. Let $\Phi(x) = f(x) - x$, so that $\lim_{x\to-\infty} \Phi(x) = +\infty$ and $\lim_{x\to\infty} \Phi(x) = -\infty$. By the intermediate value theorem there exists $c \in \mathbb{R}$ with $\Phi(c) = 0$, i.e. $f(c) = c$, so c is a fixed point.

Uniqueness: Suppose $f(c) = c$ and $f(d) = d$, with $c < d$, say. Then

$$c = f(c) > f(d) = d$$

since f reverses order; but this inequality contradicts the previous inequality $c < d$, so indeed there can be only one fixed point.

Question 4. Let $\mathcal{H}$ denote the collection of compact subsets of $\mathbb{R}$. Let $\Phi : \mathcal{H} \to \mathcal{H}$ be the iterated function system defined by the two maps $\phi_1(x) = x/10$ and $\phi_2(x) = (x+3)/10$, and let C_k denote $\Phi^k([0,1])$ for $k \geq 0$.

(a) [5 marks] For $A, B \in \mathcal{H}$, how is the *Hausdorff distance* $h(A,B)$ defined?

(b) [4 marks] Write down the sets C_1 and C_2.

(c) [5 marks] Compute $h(C_1, C_2)$.

(d) [3 marks] If C_k is expressed as a disjoint union of N_k closed intervals, compute the number N_k.

(e) [3 marks] What is the common length of each of the N_k closed intervals whose disjoint union equals C_k?

(f) [5 marks] Given a set $A \subset \mathbb{R}$, how is its *box dimension* defined?

(g) [5 marks] Using your answers to parts (d) and (e), or otherwise, show that if the box dimension of $C = \cap_{k=0}^{\infty} C_k$ exists then it must equal $\log 2/\log 10$.

(h) [5 marks] Give a description of the members of C in terms of the digits of their decimal expansion.

(i) [5 marks] If $f : C \to C$ is defined by $f(x) = 10x \pmod 1$ then find a point $x \in C$ which has minimal period 3 under f.

Solution:

(a) [5 marks] Let $d(\cdot,\cdot)$ be the usual distance function on $\mathbb{R}$. For $A \in \mathcal{H}$, and $x \in \mathbb{R}$, define $\varrho(x, A) = \min_{y \in A} d(x,y)$.

Then define $h_{BA} = \max_{x \in B} \varrho(x, A)$, and finally set

$$h(A,B) = \max(h_{AB}, h_{BA}) .$$

(b) [4 marks] $C_1 = [0, 1/10] \cup [3/10, 4/10]$, and

$$C_2 = [0, 1/100] \cup [3/100, 4/100] \cup [3/10, 31/100] \cup [33/100, 34/100].$$

(c) [5 marks] If $A = C_1$, $B = C_2$ then $h_{BA} = 0$ since $B \subset A$, whilst

$$h_{AB} = \max_{x \in C_1} \varrho(x, C_2) = \varrho(1/10, C_2) = \varrho(1/10, 4/100) = 6/100 = 3/50 ,$$

so

$$h(C_1, C_2) = \max(3/50, 0) = 3/50 .$$

(d) [3 marks] $N_k = 2^k$ because $N_0 = 1$ and the recursive procedure doubles the number of intervals at each step.

(e) [3 marks] The length is $1/10^k$, because the length of the closed intervals decreases by a factor of 10 at each step, and the length of $C_0 = [0,1]$ is 1.

(f) [5 marks] For $\varepsilon > 0$ let $N(\varepsilon)$ denote the smallest number of length-ε intervals needed to cover A. The box dimension of A is then

$$\lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{-\log \varepsilon},$$

provided the limit exists.

(g) [5 marks] If $\varepsilon_k = 1/10^k$ then $N(\varepsilon_k) = 2^k$, by parts (d) and (e), and so the box dimension equals

$$\lim_{k \to \infty} \frac{\log N(\varepsilon_k)}{-\log \varepsilon_k} = \lim_{k \to \infty} \frac{k \log 2}{k \log 10} = \frac{\log 2}{\log 10}.$$

(h) [5 marks] C consists of those numbers between 0 and 1 which have a decimal expansion whose digits all equal either 0 or 3.

(i) [5 marks] There are six such points, namely

$1/333 = 0.003003003\ldots$, $10/333 = 0.030030030\ldots$, $100/333 = 0.300300300\ldots$

$11/333 = 0.033033033\ldots$, $101/333 = 0.303303303\ldots$, $110/333 = 0.330330330\ldots$

End of Paper.