Queen Mary
University of London

B. Sc. Examination by course unit 2014

MTH6107 Chaos & Fractals (SOLUTION SHEET)

Duration: 2 hours

Date and time: 30th April 2014, 14.30–16.30

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You should attempt all questions. Marks awarded are shown next to the questions.

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Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Examiner(s): O. Jenkinson

Question 1 (a) Suppose we are given a non-empty set Σ and a map $f : \Sigma \to \Sigma$.

(i) [1 mark] What does it mean to say that $x \in \Sigma$ is a *fixed point* for f?

(ii) [2 marks] What does it mean to say that $x \in \Sigma$ is a *periodic point* for f?

(iii) [1 mark] How is the *minimal period* of a periodic point defined?

(iv) [2 marks] What does it mean to say that $x \in \Sigma$ is an *eventually periodic point* for f?

(v) [6 marks] Prove that if f is invertible then every eventually periodic point is a periodic point.

(b) [5 marks] Give a detailed statement of Sharkovsky's Theorem.

(c) Suppose the map $f : [0,1] \to [0,1]$ is defined by

$$f(x) = \begin{cases} x + 1/2 & \text{for } x \in [0, 1/2) \\ 2 - 2x & \text{for } x \in [1/2, 1] \,. \end{cases}$$

(i) [3 marks] For this map f, determine all its fixed points.

(ii) [4 marks] For this map f, determine an eventually periodic point which is not periodic.

(iii) [4 marks] For this map f, determine all its points of minimal period 2.

Solution:

(a) (i) [1 mark] It means that $f(x) = x$.

(ii) [2 marks] It means that $f^n(x) = x$ for some $n \in \mathbb{N}$.

(iii) [1 mark] It is the smallest natural number n such that $f^n(x) = x$.

(iv) [2 marks] It means that for some $m \geq 0$, the point $f^m(x)$ is a periodic point.

(v) [6 marks] Suppose that x is eventually periodic, so there exists $m \geq 0$ such that $f^m(x)$ is a period-n point for some $n \geq 1$, i.e. $f^n(f^m(x)) = f^m(x)$. In other words, $f^m(f^n(x)) = f^m(x)$. But invertibility allows us to apply $f^{-m} = (f^{-1})^m$ to both sides of this equality, giving $f^n(x) = x$, so in fact x is periodic.

(b) [5 marks] Sharkovsky's ordering $\prec$ of the natural numbers is given by:

$$\begin{gathered} 1 \prec 2 \prec 2^2 \prec 2^3 \prec \cdots \prec 2^m \prec \cdots \\ \vdots \\ \cdots \prec 2^k(2n-1) \prec \cdots \prec 2^k \cdot 7 \prec 2^k \cdot 5 \prec 2^k \cdot 3 \prec \cdots \\ \vdots \\ \cdots \prec 2(2n-1) \prec \cdots \prec 2 \cdot 7 \prec 2 \cdot 5 \prec 2 \cdot 3 \prec \cdots \\ \cdots \prec 2n-1 \prec \cdots \prec 7 \prec 5 \prec 3 \,. \end{gathered}$$

Sharkovsky's Theorem then says that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, and has a periodic orbit of minimal period n, then it has a periodic orbit of minimal period m for all $m \prec n$.

(c) (i) [3 marks] The only fixed point is $x = 2/3$.

(ii) [4 marks] The point $x = 1/6$ is a pre-fixed (hence pre-periodic) point: to see this note that $f(1/6) = 1/6 + 1/2 = 2/3$, which is the fixed point.

(iii) [4 marks] The points $1/3$ and $5/6$ are the only points of minimal period 2. To find these points, note that if $x \in [0, 1/2)$ has period 2 then $f(x) = x + 1/2$ must lie in $[1/2, 1]$, so

$$x = f^2(x) = f(x + 1/2) = 2 - 2(x + 1/2) = 1 - 2x\,,$$

so $x = 1/3$; then $f(1/3) = 1/3 + 1/2 = 5/6$.

Question 2 (a) [2 marks] For a differentiable map $f : \mathbb{R} \to \mathbb{R}$, how is the *multiplier* of a periodic orbit defined?

(b) [2 marks] Write down a condition on the multiplier which guarantees that a periodic orbit is *stable* (i.e. *attractive*).

(c) Let $f_\lambda : [-1, 1] \to [-1, 1]$ be the logistic map, defined by $f_\lambda(x) = 1 - \lambda x^2$ for parameters $\lambda \in [0, 2]$.

(i) [3 marks] For $\lambda \in [0, 2)$, compute the fixed point $x^* = x^*(\lambda) \in [-1, 1]$ of f_λ.

(ii) [3 marks] Compute the multiplier of this fixed point $x^*(\lambda)$.

(iii) [2 marks] Determine the largest value λ_1 with the property that the fixed point $x^*(\lambda)$ is stable for all $\lambda \in [0, \lambda_1)$.

(iv) [6 marks] For $\lambda > \lambda_1$, determine the periodic orbit of f_λ which has minimal period 2.

(v) [4 marks] Compute the multiplier of this period-2 orbit, and determine the largest value λ_2 with the property that this orbit is stable for all $\lambda \in (\lambda_1, \lambda_2)$.

(vi) [2 marks] Briefly define what is meant by a *period-doubling bifurcation*.

(vii) [3 marks] How is the *Feigenbaum constant* δ defined?

Solution:

(a) [2 marks] If the orbit is generated by the point x, of minimal period n, the multiplier is defined to be $(f^n)'(x)$. An alternative expression (courtesy of the chain rule) is $\prod_{i=0}^{n-1} f'(f^i x)$.

(b) [2 marks] If the multiplier is strictly smaller than 1 in absolute value then the orbit is stable.

(c) (i) [3 marks] Fixed points of f_λ satisfy $\lambda x^2 + x - 1 = 0$, so $x = \frac{-1 \pm \sqrt{1+4\lambda}}{2\lambda}$, of which only

$$x^*(\lambda) = \frac{-1 + \sqrt{1 + 4\lambda}}{2\lambda}$$

belongs to $[-1, 1]$ when $\lambda \in [0, 2)$.

(ii) [3 marks] The multiplier is

$$f_\lambda'(x^*(\lambda)) = -2\lambda x^*(\lambda) = 1 - \sqrt{1 + 4\lambda}\,.$$

(iii) [2 marks] $\lambda_1 = 3/4$. This is because the multiplier is a strictly decreasing function of λ, decreasing from value 0 at $\lambda = 0$ to value -1 at $\lambda = 3/4$.

(iv) [6 marks] The period-2 points satisfy $f_\lambda^2(x) - x = 0$. But

$$f_\lambda^2(x) = 1 - \lambda(1 - \lambda x^2)^2 = -\lambda^3 x^4 + 2\lambda^2 x^2 - \lambda + 1\,,$$

so

$$f_\lambda^2(x) - x = -\lambda^3 x^4 + 2\lambda^2 x^2 - x - \lambda + 1\,.$$

But both *fixed* points are roots of this polynomial, so $\lambda x^2 + x - 1$ is a factor of this polynomial, hence we can factorise $f_\lambda^2(x) - x$ as

$$f_\lambda^2(x) - x = -(\lambda x^2 + x - 1)(\lambda^2 x^2 - \lambda x + (1 - \lambda)).$$

Therefore the points of *minimal* period 2 are the roots of $\lambda^2 x^2 - \lambda x + (1 - \lambda)$, namely

$$x_\pm(\lambda) = \frac{1 \pm \sqrt{4\lambda - 3}}{2\lambda}.$$

(v) [4 marks] The multiplier for this period-2 orbit is then the product of

$$f_\lambda'(x_+(\lambda)) = -2\lambda x_+(\lambda) = -(1 + \sqrt{4\lambda - 3})$$

and

$$f_\lambda'(x_-(\lambda)) = -2\lambda x_-(\lambda) = -(1 - \sqrt{4\lambda - 3}),$$

namely

$$1 - (4\lambda - 3) = 4 - 4\lambda = 4(1 - \lambda).$$

This multiplier decreases from value 1 at $\lambda = \lambda_1 = 3/4$ to value -1 at $\lambda = 5/4$. We therefore see that

$$\lambda_2 = 5/4.$$

(vi) [2 marks] A *period-doubling bifurcation* is the event such as occurs at $\lambda = \lambda_1$, or alternatively at $\lambda = \lambda_2$, whereby a formerly stable period-n orbit loses its stability, and a new stable period-$2n$ orbit is born.

(vii) [3 marks] If we denote by (λ_n) the sequence of parameter values at which the period-doubling bifurcations occur, the Feigenbaum constant δ can be defined by:

$$\delta = \lim_{n\to\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n}.$$

Question 3 (a) [6 marks] Define what it means for $f : \mathbb{R} \to \mathbb{R}$ to be

(i) a *homeomorphism*,

(ii) a *diffeomorphism*,

(iii) *order preserving*.

(b) [7 marks] Prove that an order preserving map $f : \mathbb{R} \to \mathbb{R}$ does not have any points of minimal period strictly larger than 1.

Solution:

(a) (i) [2 marks] A homeomorphism is a continuous bijection whose inverse map is also continuous.

(ii) [2 marks] A diffeomorphism is defined (in this module) to be a bijection such that both f and f^{-1} are C^1 maps, i.e. they are differentiable with continuous derivative.

(iii) [2 marks] It means that if $x < y$ then $f(x) < f(y)$.

(b) [7 marks] Suppose, in order to derive a contradiction, that x is periodic of minimal period $n > 1$. In particular, $f(x) \neq x$, so we can consider two cases: either $x < f(x)$ or $f(x) < x$.

If $x < f(x)$ then $f(x) < f(f(x)) = f^2(x)$ because f is order preserving, and repeating this argument we see that $f^i(x) < f^{i+1}(x)$ for all $i \geq 0$. Thus in particular $x < f^n(x)$, so $x \neq f^n(x)$, contradicting the supposition that x is n-periodic.

If on the other hand $f(x) < x$ then $f^2(x) = f(f(x)) < f(x)$ because f is order preserving, and repeating this argument we see that $f^{i+1}(x) < f^i(x)$ for all $i \geq 0$. Thus in particular $f^n(x) < x$, so $x \neq f^n(x)$, contradicting the supposition that x is n-periodic.

Question 4 (a) [4 marks] Let $C_0 = [0,1]$. In the standard construction of the Cantor ternary set $C = \cap_{k=0}^{\infty} C_k$, describe briefly how the sets C_k are defined.

(b) [2 marks] Write down the sets C_1 and C_2.

(c) [2 marks] If C_k is expressed as a disjoint union of N_k closed intervals, compute the number N_k.

(d) [2 marks] What is the common length of each of the N_k closed intervals whose disjoint union equals C_k?

(e) [4 marks] Given a set $A \subset \mathbb{R}$, how is its *box dimension* defined?

(f) [4 marks] Let $\mathcal{H}$ denote the collection of compact subsets of $\mathbb{R}$. For $A, B \in \mathcal{H}$, how is the *Hausdorff distance* $h(A, B)$ defined?

(g) [4 marks] Compute $h(C_1, C_2)$.

(h) [4 marks] Using your answers to parts (c) and (d), or otherwise, show that if the box dimension of the ternary Cantor set $C \subset \mathbb{R}$ exists then it must equal $\log 2/\log 3$.

(i) [3 marks] Given two maps $\phi_1 : \mathbb{R} \to \mathbb{R}$ and $\phi_2 : \mathbb{R} \to \mathbb{R}$, how is the corresponding *iterated function system* $\Phi : \mathcal{H} \to \mathcal{H}$ defined?

(j) [3 marks] Write down two maps $\phi_1 : \mathbb{R} \to \mathbb{R}$ and $\phi_2 : \mathbb{R} \to \mathbb{R}$ such that the ternary Cantor set C is the fixed point of the corresponding iterated function system Φ.

Solution:

(a) [4 marks] The set C_{k-1} is a disjoint union $\cup_i I_i$ of closed intervals. If from each of these closed intervals I_i we remove the 'open middle third', we are left with a pair of closed intervals I_i^- and I_i^+, each of length a third the length of I. The union $\cup_i (I_i^- \cup I_i^+)$ of these intervals is then defined to be the set C_k.

(b) [2 marks] $C_1 = [0, 1/3] \cup [2/3, 1]$, and

$$C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1].$$

(c) [2 marks] $N_k = 2^k$ because $N_0 = 1$ and the recursive procedure doubles the number of intervals at each step.

(d) [2 marks] The length is $1/3^k$, because the length of the closed intervals decreases by a factor of 3 at each step, and the length of $C_0 = [0, 1]$ is 1.

(e) [4 marks] For $\varepsilon > 0$ let $N(\varepsilon)$ denote the smallest number of length-ε intervals needed to cover A. The box dimension of A is then

$$\lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{-\log \varepsilon},$$

provided the limit exists.

(f) [4 marks] Let $d(\cdot,\cdot)$ be the usual distance function on $\mathbb{R}$. For $A \in \mathcal{H}$, and $x \in \mathbb{R}$, define $\varrho(x, A) = \min_{y \in A} d(x, y)$.

Then define $h_{BA} = \max_{x \in B} \varrho(x, A)$, and finally set

$$h(A, B) = \max(h_{AB}, h_{BA})\,.$$

(g) [4 marks] If $A = C_1$, $B = C_2$ then $h_{BA} = 0$ since $B \subset A$, whilst

$$h_{AB} = \max_{x \in C_1} \varrho(x, C_2) = \varrho(1/6, C_2) = 1/6 - 1/9 = 1/18\,,$$

so

$$h(C_1, C_2) = \max(1/18, 0) = 1/18\,.$$

(h) [4 marks] If $\varepsilon_k = 1/3^k$ then $N(\varepsilon_k) = 2^k$, by parts (c) and (d), and so the box dimension equals

$$\lim_{k \to \infty} \frac{\log N(\varepsilon_k)}{-\log \varepsilon_k} = \lim_{k \to \infty} \frac{k \log 2}{k \log 3} = \frac{\log 2}{\log 3}\,.$$

(i) [3 marks] It is defined by $\Phi(A) = \cup_{i=1}^{2} \phi_i(A)$ for all $A \in \mathcal{H}$.

(j) [3 marks] We may take $\phi_1(x) = x/3$ and $\phi_2(x) = (x + 2)/3$.

End of Paper