Queen Mary
University of London

Final Exam: 6th Jan

2.5 Hours 10:00 — 12:30 pm

Group Theory

UG

Week 12, Lecture 1, 2 & 3

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3 Hours 10:00 — 1:00 pm

MSc

LH-TM-KO

(CAMPUS-M)

Table of Contents

Revision:

Suppose G is a finite group and p a prime.

$|G| = p^a b, \quad p \nmid b.$

A Sylow p-subgroup of G is a subgroup of order p^a.

Examples:

1) $G = C_{100}$

$\langle g^{25} \rangle \leq G$ is a Sylow 5-Subgroup.

2) $G = S_3$

$\langle (1\,2) \rangle$ is a Sylow 2-Subgroup

$\langle (1\,2\,3) \rangle$ is a Sylow 3-Subgroup.

Sylow's Theorem 1

Theorem: Suppose G is a finite group and p is a prime. Then G has at least one Sylow p-subgroup.

Sol $|G| = p^a b$ $Y = \text{orb}(X)$

$$X = \{X_1, \ldots X_n, \ldots\}$$

$$|X_1| = p^a = |X_2| = \ldots = |X_n| = p^a$$

$$\pi_g(s_1, s_2, \ldots, s_{p^a}) = (gs_1, gs_2, \ldots, gs_{p^a})$$

$$P = \text{Stab}(S)$$

Then P is Sylow p-subgroup.

Examples

Let $G = \mathcal{U}_9 = \{1, 2, 4, 5, 7, 8\}$, and $p = 3$. G is small enough that we can easily find a subgroup of order 3, but let's follow the proof of Sylow's Theorem 1.

Solution:

$$U_9 = \{1, 2, 4, 5, 7, 8\}$$

$$P = \{1, 4, 7\}$$

is a Sylow 3-Subgroup

Classification of p-Groups

Notation: Suppose G a finite group. Let $\mathrm{Syl}_p(G)$ denote the set of Sylow p-subgroups of G, and let $\mathrm{n}_p\mathrm{G}$ denote the number of Sylow p-subgroups of G.

$$n_p(G) = 1 + np \mid |G|$$

Classification of p-Groups

$10 = 2 \times 5$

Take $G = \mathcal{D}_{10}$. Then a Sylow 5-subgroup is a subgroup of order 5.
One such subgroup is $\langle r \rangle$. In fact, this is the only example: the elements of $\mathcal{D}_{10} \setminus \langle r \rangle$ all have order 2, so cannot be contained in a subgroup of order 5. So a subgroup of order 5 is contained in $\langle r \rangle$, so must be $\langle r \rangle$. So $\mathrm{n}_2(\mathcal{D}_{10}) = 1$.
As a special case: if P is the only Sylow p-subgroup of G, then $P \trianglelefteq G$.

Sol $\mathcal{D}_{10} = \{1, r, r^2, r^3, r^4, s, rs, r^2s, r^3s, r^4s\}$

$n_5 = 1 + 5k \mid 10$

$n_2 = 1 + 2k \mid 10$

$n_5 = 1$

$n_2 = 1, \quad k = 0$

Sylow 2-subgroup of $\mathcal{D}_{10}$ is one $= \{1, s\}$

Sylow 5-subgroup of $\mathcal{D}_{10}$ is one $= \{1, r, r^2, r^3, r^4\}$

Classification of p-Groups

Proposition 7.9

Suppose G is a finite group and $P, Q \in \mathrm{Syl}_p G$ with $gQg^{-1} = Q$ for every $g \in P$. Then $P = Q$.

If P & Q are Sylow p- Subgroups of a finite group G, Then P & Q are conjugate

i.e $gQg^{-1} = P$

Classification of p-Groups

Sylow's Theorem 2, 7.10

Suppose G is a finite group and p is a prime. Then all the Sylow p-subgroups of G are conjugate.

Sylow's Theorems

Sylow's Theorem 3, 7.11

Suppose G is a finite group, and p is a prime, and write $|G| = p^a b$, where $p \nmid b$. Then $\mathrm{n}_p(G) \equiv 1 \bmod p$, and $\mathrm{n}_p(G) \mid b$.

$n_p(G) = 1 + pk \,\Big|\, |G|$

$k = 0, 1, 2, \ldots$

& we choose the smallest such k

Sylow's Theorems

Remark

Sylow's Theorem 2 shows that if $P \in \mathrm{Syl}_p(G)$ and $P \trianglelefteq G$, then P is the only Sylow p-subgroup of G (because any other Sylow p-subgroup would have to be conjugate to P). In particular, if G is abelian, then (since all subgroups of an abelian group are normal) G has a unique Sylow p-subgroup.

Simple: G is called Simple, if only Normal Subgroup are $\{e\}$ & G.

Sylow's Theorems

Example

We can show that C_{15} is the only group of order 15 up to isomorphism.

Sol

$|G| = 3 \times 5$

$\left.\begin{array}{l}\exists \text{ a Sylow } 3\text{-Subgroup} \\ \exists \text{ a Sylow } 5\text{-Subgroup}\end{array}\right\}$ By Sylow Theorem 2.

To check how many such Sylow Theorem exist, we use Sylow Theorem 3.

Sylow's Theorems

$n_3 = 1 + 3k \mid 15 \quad k = 0, 1, 2, \ldots$

$n_3 = 1$, there is only one Sylow 3-Subgroup

$\langle g^5 \rangle = \{1, g^5, g^{10}\}$

$n_5(G) = 1 + 5k \mid 15, \quad k = 0, 1, 2, \ldots$

$k = 0$

$n_5 = 1 \quad \exists$ only one Sylow 5-Subgroup

$\langle g^3 \rangle = \{g^3, g^6, g^9, g^{12}, g^{15} = 1\}$

Elements of $G = \{1, g, g^2, g^3, g^4, g^5, g^6, \ldots, g^{14}\}$

Sylow's Theorems

have order 1, 3, 5 or 15.

when g is of order 3 or 5 we get Sylow 3-Subgroup or 5-Subgroup.

If g is of order 15, then

$$\langle g \rangle = \{1, g, g^2, \ldots, g^{15}\} = C_{15}.$$

So C_{15} is the only group of order 15.

Sylow's Theorems

$|G| = 2^2 \times 5$

Examples:

Suppose G is a group of order 20; then we claim that G cannot be simple.

$n_5(G) = 1 + 5k \mid 20$

$n_5(G) = 1$

G has Sylow 2-Subgroup

G has Sylow 5-Subgroup

Suppose P is the Sylow 5-Subgroup

Since P is the only Sylow 5-Subgroup

$gPg^{-1} = P$, P is Normal

So G is not Simple.

Sylow's Theorems

$|G| = 3 \times 2^2 = 12$

Example: For a more complicated example, suppose G is a group of order 12; again we claim that G cannot be simple.

Sylow 3-Subgroup

Sylow 2-Subgroup

$n_3(G) = 1 + 3k \mid 12$

$k = 1$

$n_3(G) = 4$

$n_2(G) = 1 + 2k \mid 12,$ $k = 1$

$n_2(G) = 1 + 2 = 3$

$n_3(G) = 1, 4$

$n_2(G) = 1, 3$

$n_3 = 4,\ n_2 = 3$

$n_3 = 1,\ n_2 = 1$ — Not simple

$n_3 = 4,\ n_2 = 1$ — Not Simple

$n_3 = 1,\ n_2 = 3$ — Not Simple

$n_3 = 4, \quad n_2 = 3$

$\{H_1, H_2, H_3, H_4\}$

4 Sylow 3- Subgroup of order 3

3, 2, 2, 2

$\{\overset{4}{K_1}, \overset{3}{K_2}, \overset{3}{K_3}\}$

of 3 Subgroup of order 4

Total elements

$= 9 + 10 = 19 > 12$

Not possible

So 12 cannot be simple

Question: G is a group of order 56.

$|G| = 2^3 \times 7$

$\exists$ Sylow 2-Subgroup

$\exists$ Sylow 7- Subgroup

$n_2(G) = 1, 7$ Subgroups of order 8

$n_7(G) = 1, 8$ Subgroups of order 7

n_2 : 1 , 7

n_7 : 1 , 8

G is

$n_2 = 1, n_7 = 1$ Not Simple

$n_2 = 1, n_7 = 8$ Not Simple

$n_2 = 7, n_7 = 1$ Not Simple

Lets discuss $n_2 = 7$, $n_7 = 8$

$\left\{ \overset{8}{H_1}, \overset{7}{H_2}, \overset{7}{H_3}, \overset{7}{H_4}, \overset{7}{H_5}, \overset{7}{H_6}, \overset{7}{H_7} \right\}$,
of order 8

$\left\{ \overset{7}{K_1}, \overset{6}{K_2}, \ldots, \overset{6}{K_8} \right\}$
of order 7

$7 \times 6 + 8$
$= 42 + 8 = 50$

$7 + 6 \times 7$
$49 = 7 + 42$

$50 + 49 = 99 > 56$ Not possible

So $|G| = 56$ Cannot be Simple

Example: Group of order 24 cannot be Simple.

$24 = 2^3 \cdot 3$

Sylow 2-Subgroups

Sylow 3-Subgroup

$n_2(G) = 1 + 2k \mid 24$

$n_2 = 1, 3$

$n_3(G) = 1 + 3k \mid 24$

$n_3 = 1, 4$

$n_2 = 1, 3$

$n_3 = 1, 4$

$n_2 = 1, n_3 = 1$ Not Possible

$n_2 = 1, n_3 = 4$ Not Possible

$n_2 = 3, n_3 = 1$ Not Possible

$n_2 = 3, n_3 = 4$ Lets check this

$$\begin{Bmatrix} H_1, & H_2, & H_3 \\ 8 & 7 & 7 \end{Bmatrix} \qquad \begin{Bmatrix} K_1, & K_2, & K_3, & K_4 \\ 3 & 2 & 2 & 2 \end{Bmatrix}$$

$22 \qquad\qquad 9$

$22 + 9 = 31 > 24$ Not possible

A group of order 24 cannot be simple.

Style of Final Exam Paper

Question I:

* Give examples of groups
* Not isomorphic to each other
* Caylay Table, Inverse of elements & identity elements uniqueness,
Abelian, Subgroups of $\leq GL_2(\mathbb{R})$

Question 4

* Composition Series, C_{15}, D_8, S_4
* Group can be simple or not $n_p(G)$
* Inner Automorphism
* Outer Automorphism

Question 2:

- permutations, cycles, orders
- A_n or not, types of cycles, conjugacy classes, Centre.
- $Z(S_n)$, Centraliser
- D_8, D_{10}, D_{12}
- commutator

Question 3

- Isomorphism Theorems
- Centraliser, Normaliser
- Group Actions,
- Orbits, Stabilisers

Sylow's Theorems

Complete the following table, so as to obtain the Cayley Table of a group.

	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	b	1	d	e	c
b	b	1	a	e	c	d
c	c	e	d	1	b	a
d	d	c	e	a	1	b
e	e	d	c	b	a	1

Sylow's Theorems

$-i(1)i = -(-1) = 1$

$$Q_8 = \{1, i, j, k, -i, -j, -k, -1\}$$

$ji(-j) = -i$

$k \cdot i (-k)$

$j(-k) =$

$= -j \cdot k$

$= -i$

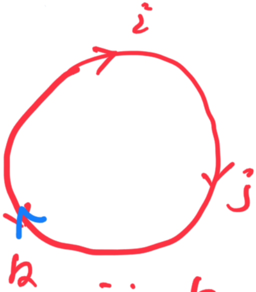
$k(1)(-k)$

$= +1$

$\{1\}, \{-1\}$

$\{j, -j\}$

$\{k, -k\}$



$i \cdot j = k$

$j \cdot k = i$

$k \cdot i = j$

Sylow's Theorems

	1	a	b	c	d
1	$\cancel{\boxed{1}}$	a	b	c	~~d~~
a	a		–		$\boxed{1}$
b	b	$\boxed{1}$	c	~~d~~	a
c	c		$\boxed{1}$		
d	d		a	$\boxed{1}$	c

$ad = 1, \quad ba = 1$

Write Reference to the axioms why the table cannot be completed so as to obtain the Cayley table of a group.

Exams Style Questions

$cb = 1, \quad bc = d$

Inverses are unique

Cannot be a group.

Exams Style Questions

Group Action:

$$G' = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R},\ a \neq 0 \right\} \leq GL_2(\mathbb{R}$$

G' defines an action on $\mathbb{R}$

$$\pi_g(x) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \underset{\mathbb{R}}{\underset{\cap}{(x)}} = ax + b \in \mathbb{R}$$

$$G \times \mathbb{R} \longrightarrow \mathbb{R}$$

prove that this is

A1) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(x) = 1 \times x + 0 = x$

$\pi_e(x) = x$ identity

A2) $\begin{array}{cc} \pi_g & \pi_f \\ \begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} a_2 & b_2 \\ 0 & 1 \end{pmatrix} \end{array}(x) = \underbrace{a_2 x + b_2}_{\pi_g \pi_f(x)}$

$= a_1 a_2 x + a_1 b_2 + b_2$ — (1)

$\pi_{gf}(x) = \begin{pmatrix} a_1 a_2 & a_2 b_2 + b_1 \\ 0 & 1 \end{pmatrix}(x)$

QMplus Quiz

$$= a_1 a_2 x + a_1 b_2 + b_2 \quad \text{— } (2)$$

$$(1) = (2)$$

$$\lambda_g \lambda_f(n) = \lambda_{gf}(n)$$

Week-11 & Week 12 QMplus page

Discuss the orbit & Stabiliser of $\{1\}$

$$\{\lambda_g(1)\}$$

$$\boxed{\lambda_g(1) = 1}$$

Week-10 Slides

Question: 1) Composition Series of S_4.

$$S_4 \rhd A_4 \rhd V_4 \rhd C_2 \rhd \{1\}$$

2) $$C_{15} \rhd \langle g^3 \rangle \rhd \{1\}$$

$$C_{15} \rhd \langle g^5 \rangle \rhd \{1\}$$

Conjugacy Overview

$H \leq G$, the conjugate subgroup of H is

$$gHg^{-1} = \{ghg^{-1} \mid h \in H\}$$

H is Normal $\iff gHg^{-1} = H$

Remark: $ccl(e) = \{e\}$

as $geg^{-1} = e$

when to compute the conjugacy class of $\{x\}$ you need to compute all the elements which do not commute with x.

as if $gxg^{-1} = x$ $\quad ccl_G(x) = \{x\}$

$gx = xg \iff x$ is is $Z(G)$

Lemma: conjugacy is an equivalence relation.

- Reflection
- Symmetric
- Transitive

Consider D_8

$$ccl_{D_8}(r) = \{grg^{-1} \mid g \in D_8\} = \{r, r^3\}$$

$$srs^{-1} = sr \cdot s = r^{-1} = r^3$$

$$rs \cdot r(rs)^{-1} = rs \cdot r \cdot s^{-1} \cdot r^{-1}$$

$$= rr^{-1}s \cdot s \cdot r^{-1}$$

$$= r^{-1} = r^3$$

$$ccl_{D_8}(s) = \{grg^{-1} \mid g \in D_8\} = \{r^2 s, s\}$$

$$rsr^{-1} = rs \cdot r^3$$
$$= rs \cdot r r^2$$
$$= r r^{-1} s \cdot r^2$$
$$= s \cdot r^2 = r^2 s$$

$$ccl_{D_8}(r^2) = \{e, r^2\}$$

$$ccl_{D_8}($$

Conjugacy in D_{10}

$$ccl_{D_{10}}(r^3) = \{ g r^3 g^{-1} \mid g \in D_{10} \}$$

$$= \{ r^3, r^2 \}$$

Revose: Every Normal Subgroup is the union of the conjugacy classes.

- G/N is a quotient group where N is a normal Subgroup
- If N is normal, all left cosets coincide with Right cosets.

Some Useful Notations

Throughout this course, we use the following notation.

- $\mathcal{C}_n$ denotes the cyclic group of order n.
- Klein group often symbolized by the letter $\mathcal{V}_4$ or as $K_4 = \mathbb{Z}_4 \times \mathbb{Z}_4$ denotes the group $\{1, a, b, c\}$, with group operation given by

$$a^2 = b^2 = c^2 = 1, \qquad ab = ba = c,\ ac = ca = b,\ bc = cb = a.$$

- $\mathcal{U}_n$ is the set of integers between 0 and n which are prime to n, with the group operation being multiplication modulo n.

Some Useful Notations

- $\mathcal{D}_{2n}$ is the group with $2n$ elements
$$1,\ r,\ r^2,\ \ldots,\ r^{n-1},\ s,\ rs,\ r^2s,\ \ldots,\ r^{n-1}s.$$
The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.
- $\mathcal{S}_n$ denotes the group of all permutations of $\{1, \ldots, n\}$, with the group operation being composition.
- $GL_n(\mathbb{R})$ is the group of $n \times n$ invertible matrices with entries in $\mathbb{R}$, with the group operation being matrix multiplication.
- $\mathcal{Q}_8$ is the group $\{1, -1, i, -i, j, -j, k, -k\}$, in which
$$i^2 = j^2 = k^2 = -1, \qquad ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$