<u>Basin of attraction</u> (revision)

Although we can discuss "basin of attraction" ~~for~~ in connection with any periodic orbit, we will focus on the case where the periodic orbit is simply a fixed point.

The concept of basin of attraction is mainly connected with <u>attracting</u> fixed points.

The definition is:

If p is a (attracting) fixed point for a map $f: \mathbb{R} \to \mathbb{R}$, its <u>basin of attraction</u> is the set

$$\text{Basin}(p) = \{x \in \mathbb{R} : \lim_{n\to\infty} f^n(x) = p\}$$

More generally, if $I \subseteq \mathbb{R}$ is an interval, and $f: I \to I$, and p is a fixed point of f, then

$$\text{Basin}(p) = \left\{ x \in \mathbb{R} : \text{the limit } \lim_{n\to\infty} f^n(x) \text{ exists and equals } p \right\}$$

<u>Remark</u>: <u>In general</u>, there is no reason to think that the limit $\lim_{n\to\infty} f^n(x)$ will exist, so we can think of x belonging to the basin of attraction as being a "special property".

<u>Example</u> If $f: \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x/2$ then $p = 0$ is a fixed point, and for any $x \in \mathbb{R}$,

$$f^n(x) = \frac{x}{2^n} \to 0 \text{ as } n \to \infty.$$

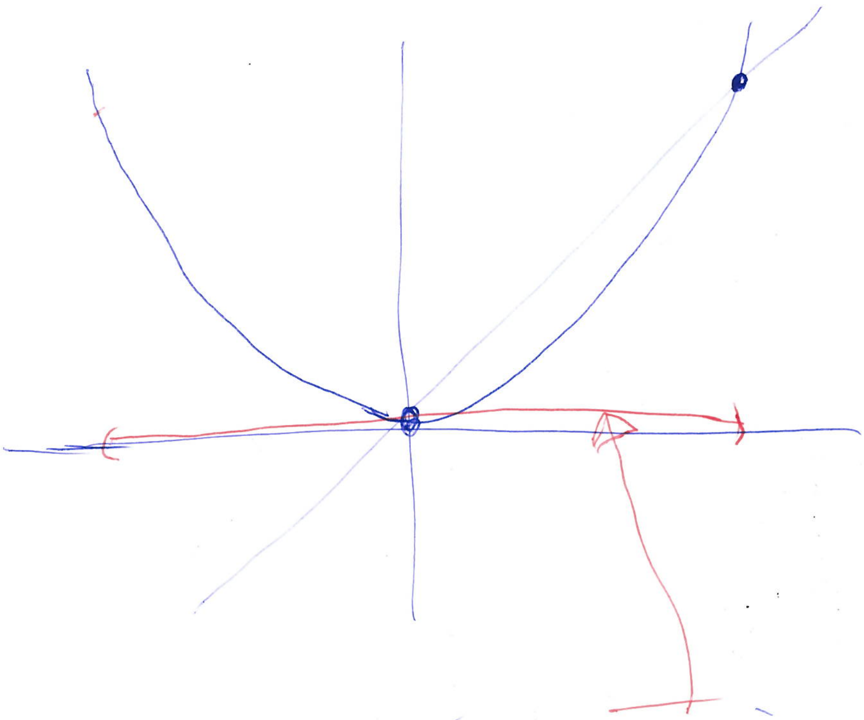
So in this case, <u>every</u> point in $\mathbb{R}$ belongs to Basin(0),

i.e. $Basin(0) = \mathbb{R}$

<u>Example</u> If $f: \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x$ then $p = 0$ is a fixed point (though it is a <u>repelling</u> fixed point, since $|f'(0)| = |3| = 3$, rather than an attracting fixed point) and since $|f^n(x)| \to \infty$ as $n \to \infty$, for all non-zero $x \in \mathbb{R}$, then the basin of attraction of this fixed point is as small as possible, in other words $Basin(p) = \{p\} = \{0\}$

Example $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

There is an attracting fixed point at $p=0$, and a repelling fixed point at $q = 1$.



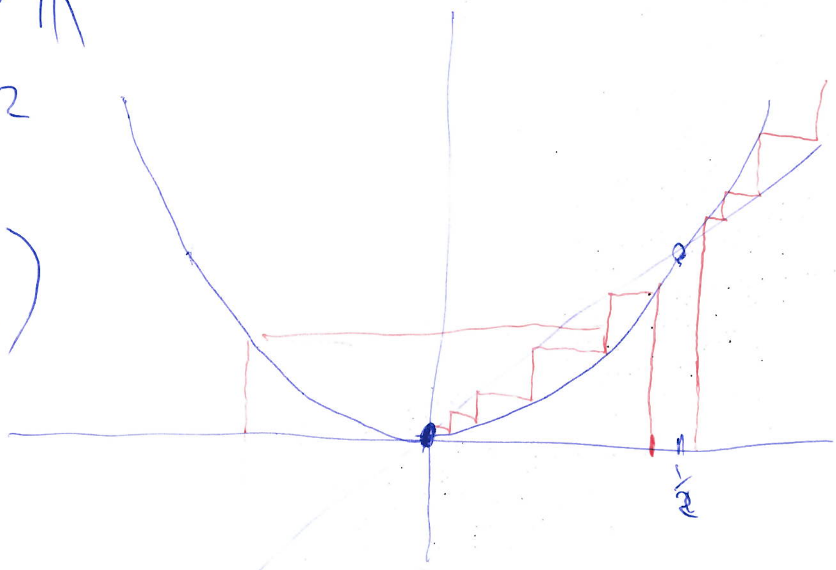
$\text{Basin}(0) = (-1, 1)$ (the open interval from -1 to 1)

($\text{Basin}(1) = \{-1, 1\}$ (i.e. a set with just 2 elements))

Example $f: \mathbb{R} \to \mathbb{R}$

$f(x) = 2x^2$

$\text{Basin}(0) = (-\tfrac{1}{2}, \tfrac{1}{2})$

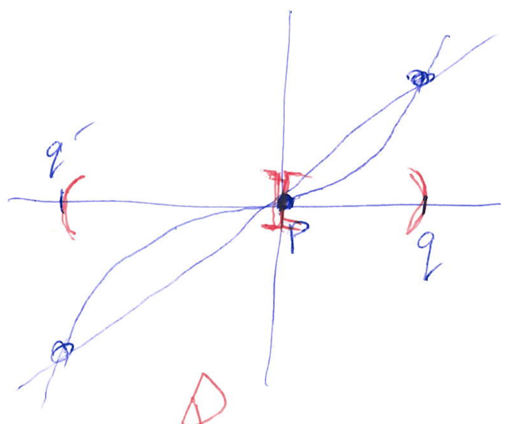


Rule of thumb

If p is an attracting fixed point, and $q > p$ is the "next largest" fixed point, and q is repelling, then the interval $[p, q)$ is in the basin of attraction of p (i.e. $[p, q) \subseteq \text{Basin}(p)$)

Similarly, if $q < p$ and is the next smallest fixed point, and is repelling, then $(q, p] \subseteq \text{Basin}(p)$.

Note: This is a rough and ready rule of thumb (needing at least that f is continuous)



Here (q', q) is in Basin(p)

Logistic maps

$f:[0,1]\to[0,1]$
$f_\mu(x)=\mu x(1-x)$

Recall: If $1<\mu<3$ then the non-zero fixed point is an attracting fixed point.

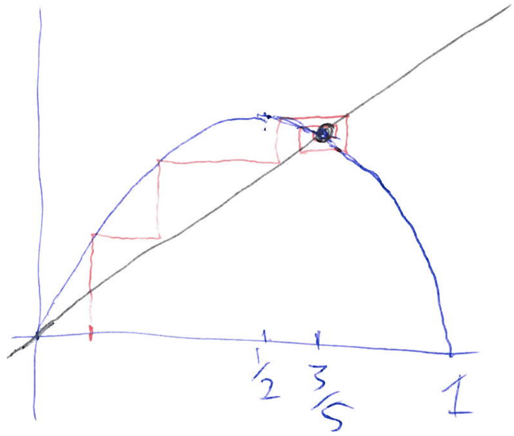
Q. What is its basin of attraction?

A. It is ~~[illegible]~~ of $(0,1]$

(i.e. the whole of $[0,1]$, except for the repelling fixed point at 0)

e.g. $f(x)=f_{5/2}(x)=\frac{5}{2}x(1-x)$

then the fixed point is at $x=3/5$.



Check (graphically) that if you pick any $x\in(0,1]$ then $f^n(x)\to 3/5$ as $n\to\infty$.

e.g. $f(x)=f_{3/2}(x)$